Classification of 3-numerical semigroups by means of L-shaped tiles

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Numerical semigroups

Numerical semigroups

S is a numerical semigroup, i.e.
an additive subsemigroup of $\mathbb{N}$ with $0 \in S$ and $g.c.d.(S) = 1$.

$\mathbb{N} - S$ is finite and we denote by
$\delta(S) = \text{Card}(\mathbb{N} - S)$ the number of *gaps* in S.

$\text{c}(S) = \text{mín}\{s \in S : \; n \geq s \Longrightarrow n \in S\}$ is
the *conductor* element of S.

S admits a minimum set of generators: $S = \langle b_0, ..., b_g \rangle$,
$b_0 = min(S - \{0\}), \;\; b_{i+1} = \text{mín}(S - \langle b_0, ..., b_i \rangle), \; i = 0, ..., g-1$.

The *Apéry set* of S with respect to b_0, is
$\text{Ap}(S, b_0) = \{s \in S : s - b_0 \notin S\}$.

Monomial curve associated to a numerical semigroup

Numerical semigroups are useful in the study of curve singularity.

If $S = \langle b_0, ..., b_g \rangle$, then $X_0 = t^{b_0}, \ldots, X_g = t^{b_g}$ is la *monomial curve* associated to S.

The embedding of the monomial curve in the affine space $\mathbf{A}_{g+1}$ is given by the K-algebra homomorphism

$$\begin{array}{ccc} \phi_0 : A = K[X_0, \ldots, X_g] & \longrightarrow & K[t^{b_0}, \ldots, t^{b_g}] \\ X_i & \longrightarrow & t^{b_i} \\ X_0^{\lambda_0} \ldots X_g^{\lambda_g} & \longrightarrow & t^{\lambda_0 b_0 + \ldots \lambda_g b_g} = t^m \end{array}$$

In this way, a problem in the semigroup is now a problem in the ideal $\ker \phi_0$ and viceversa, and

$$R = \operatorname{Im} \phi_0 = K[t^m, m \in S] = K[S]$$

is the affine coordinate algebra of the monomial curve associated to S.

Monomial curve associated to a numerical semigroup

In fact, R and A are graded rings with significant degrees only on S, in the following way

$$R = \bigoplus_{m \in S} R_m, \text{ with } R_m = Kt^m; \quad A = \bigoplus_{m \in S} A_m$$

with A_m the vector space spanned by the monomials $X_0^{\lambda_0} \dots X_g^{\lambda_g}$ with $\lambda_0 b_0 + \cdots + \lambda_g b_g = m$ and ϕ_0 being a degree zero homomorphism.

In 1991, Campillo & M. introduced combinatorics objects (*Koszul complexes*) associated to relations $m = \lambda_0 b_0 + \cdots + \lambda_g b_g$, with $\lambda_i \in \mathbb{N}$, for the elements of S in order to get information from them.

Classes of numerical semigroups

Each class of numerical semigroups that we will consider in wath follows receives the name of its corresponding class of associated curves.

Any S verifies $m \in S \Longrightarrow c(S) - 1 - m \notin S$.

S is *symmetric* if it satisfies $m \in S \Longleftrightarrow c(S) - 1 - m \notin S$.

If S is symmetric, $c(S) = 2\delta(S)$
(Frőberg, Gotlieb & Hǻggkvist 1997).

Symmetric semigroups are exactly those for which the K-algebra R is *Gorenstein* (Campillo & M. 1991).

Classes of numerical semigroups

S is a *complete intersection* when the graded ring $R = K[S]$ is a complete intersection, i.e. when the ideal $\ker \phi_0$ can be generated by g homogeneous elements.

In this case, we can write the associated monomial curve $X_i = t^{b_i}$, $0 \leq i \leq g$, as an intersection of g hypersurfaces in the way $f_i(X_0, \ldots, X_g) = 0, 1 \leq i \leq g$.

Classes of numerical semigroups

Let $N_i = e_{i-1}/e_i$, where $e_i = g.c.d.(b_0, ..., b_i)$, $i = 0, 1, ..., g$.

S is *free* if it verifies $N_i b_i \in \langle b_0, \ldots, b_{i-1} \rangle$, $i = 1, \ldots, g$.

Herzog 1970 proved that free semigroup implies symmetric semigroup. It also implies complete intersection, since if $N_i b_i = \lambda_{i0} b_0 + \cdots + \lambda_{i,i-1} b_{i-1}$, with $\lambda_{ik} \in \mathbb{N}$, then $t^{N_i b_i} = t^{\lambda_{i0} b_0 + \cdots + \lambda_{i,i-1} b_{i-1}}$ and, since $X_i = t^{b_i}$,
the looked hypersurfaces are
$f_i(X_0, \ldots, X_g) = X_i^{N_i} - X_0^{\lambda_{i0}} \cdots X_{i-1}^{\lambda_{i,i-1}} = 0$, $1 \leq i \leq g$.

We can obtain the monomial curve from the equations $f_i = 0$ by recurrence from $i = 1$ to $i = g$.

Classes of numerical semigroups

S is a *plane curve semigroup* if it is free and
$N_i b_i < b_{i+1}, \ \ i = 1, \ldots, g-1$.

To each plane curve with parametric representation
$x = t^{c_0}, \ y = t^{c_1} + \cdots + t^{c_g}$,
we can associate a semigroup $S = \langle b_0, \ldots.., b_g \rangle$ where
$b_0 = c_0$, $b_1 = c_1$ and $b_{i+1} = c_{i+1} - c_i + N_i b_i$.

If $S = \langle b_0, \ldots.., b_g \rangle$ is a plane curve semigroup, then
$\mathrm{Ap}(S, b_0) = \{\mu_1 b_1 + \cdots + \mu_g b_g : \ 0 \le \mu_k < N_k, 1 \le k \le g\}$
and the Apéry terms only have this form.

Classes of numerical semigroups

In general:

$$\text{plane curve} \Longrightarrow \text{free} \Longrightarrow \begin{cases} \text{complete intersection} \\ \text{symmetric} \end{cases}$$

For $S = \langle b_0, b_1, b_2 \rangle$:

$$\text{plane curve} \Longrightarrow \text{free} \Longrightarrow \text{complete intersection} \Longleftrightarrow \text{symmetric}$$

For $S = \langle b_0, b_1 \rangle$:

$$\begin{cases} S \text{ is a plane curve with} \\ \mathrm{Ap}(S, b_0) = \{0, b_1, 2b_1, \ldots, (b_0 - 1)b_1\} \text{ and} \\ c(S) = (b_0 - 1)(b_1 - 1) \end{cases}$$

L-shapes related to a 3-numerical semigroup

Double–loop digraph

A *double–loop digraph* $G = G(N; a, b)$, with $1 < a < b < N$ and $gcd(a, b, N) = 1$, has

$$\begin{aligned} V(G) &= \mathbb{Z}_N = \{[0]_N, [1]_N, ..., [N-1]_N\}, \\ A(G) &= \{[i]_N \xrightarrow{a} [i+a]_N, [i]_N \xrightarrow{b} [i+b]_N : \ i = 0, ..., N-1\}. \end{aligned}$$

The weight of a path in G is the sum of the weights of its arcs.

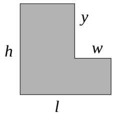
A path from $[i]_N$ to $[j]_N$ is minimum if it has minimum weight p_{ij} for all paths from $[i]_N$ to $[j]_N$.

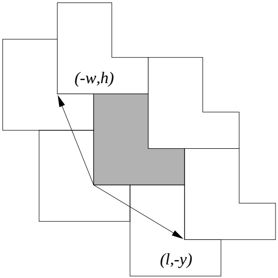
It is also known as the distance $\mathrm{d_G}([i]_N, [j]_N) = p_{ij}$ in G.

Double–loop digraph

Each unitary square $[\![i,j]\!] := [i,i+1] \times [j,j+1] \in \mathbb{R}^2$, with $(i,j) \in \mathbb{Z}^2$, is associated to the vertex $[ia+jb]_N$ reached by a path of length $ia+jb$ from the vertex $[0]_N$.

The plane is periodically tessellated by L-shaped tiles: $L(l,h,w,y)$.





Minimum distance diagrams

Each L-shape $L(l, h, w, y)$ has $N = lh - wy$ unitary squares that represent the N vertices of the digraph.

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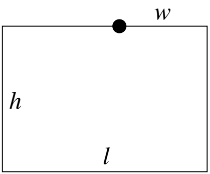
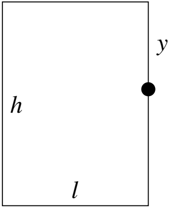
The L-shape $L(l,h,w,y)$ is *degenerated* (rectangular) if $wy = 0$.



L(l,h,w,0)



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Minimum distance diagrams

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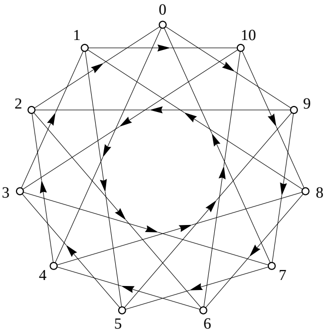
Among all possible plane tessellations related to the digraph $G(N; a, b)$, we are interested in those that are *minimum distance diagrams* (MDD), $\mathcal{H}$, i.e. if $[\![i, j]\!] \in \mathcal{H}$, then $\mathrm{d_G}([0]_N, [ia + jb]_N) = ia + jb$.

In this way, $\{ia + jb : \ [\![i, j]\!] \in \mathcal{H}\} = \mathrm{Ap}(\langle a, b, N\rangle, N)$.

Definition: An L-shape $\mathcal{H}$ is related to a semigroup $S = \langle a, b, N\rangle$ if $\mathcal{H}$ is a MDD of the digraph $G(N; a, b)$.

Example MDD

Example $G(11; 4, 9)$: $\mathcal{H} = L(5, 3, 4, 1)$



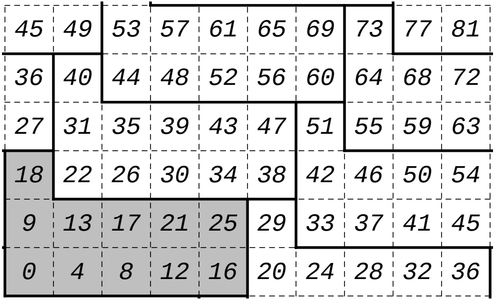
4	8	1	5	9	2	6	10	3	7
6	10	3	7	0	4	8	1	5	9
8	1	5	9	2	6	10	3	7	0
10	3	7	0	4	8	1	5	9	2
1	5	9	2	6	10	3	7	0	4
3	7	0	4	8	1	5	9	2	6
5	9	2	6	10	3	7	0	4	8
7	0	4	8	1	5	9	2	6	10
9	2	6	10	3	7	0	4	8	1
0	4	8	1	5	9	2	6	10	3

Each unitary square $[\![i, j]\!] \in \mathbb{N}^2$ has the label $[4i + 9j]_{11}$

related L-shape

Example: $S = \langle 4, 9, 11 \rangle$, $\mathcal{H} = L(5, 3, 4, 1)$

$\mathrm{Ap}(S, 11) = \{0, 4, 8, 9, 12, 13, 16, 17, 18, 21, 25\}$



Each unitary square $[\![i, j]\!] \in \mathbb{N}^2$ has the label $4i + 9j$

Previous results

Previous results

Each 3-numerical semigroup has at least one related L-shape (Rödseth 1978).

Theorem (A. & M. 2010) Let $\mathcal{H} = L(l,h,w,y)$ be related to $S = \langle a,b,N\rangle$. We have

- If $(la-yb)(hb-wa) > 0,$ then there is no other tile related to S.
- If $hb = wa$, then S also has related the L-shape

$$\mathcal{T}_1(\mathcal{H}) = \begin{cases} L(w, 2h-y, 2w-l, h) & l < 2w, \\ L(w, (\lfloor l/w \rfloor + 1)h - y, w-r, h) & l > 2w > 0, \\ & l = \lfloor l/w \rfloor w + r, \\ & 0 < r < w, \\ L(w, lh/w - y, 0, h) & l \geq 2w > 0, w \mid l. \end{cases}$$

Previous results

- If $la = yb$, then S also has related the L-shape

$$\mathcal{T}_2(\mathcal{H}) = \begin{cases} L(2l - w, y, l, 2y - h) & h < 2y, \\ L((\lfloor h/y \rfloor + 1)l - w, y, l, y - r) & h > 2y > 0, \\ & h = \lfloor h/y \rfloor y + r, \\ & 0 < r < y, \\ L(lh/y - w, y, l, 0) & h \geq 2y > 0, y \mid h. \end{cases}$$

- $\mathcal{T}_2 \circ \mathcal{T}_1(\mathcal{H}) = \mathcal{H}$ and $\mathcal{T}_1 \circ \mathcal{T}_2(\mathcal{H}) = \mathcal{H}$.
- $\mathcal{T}_1(\mathcal{H}) \neq \mathcal{H}$ y $\mathcal{T}_2(\mathcal{H}) \neq \mathcal{H}$.
- If $(la - yb)(hb - wa) = 0$, then there are exactly two tiles related to S.

Previous results

Theorem (A. & M. 2010): Let $S = \langle a, b, N \rangle$ be a semigroup and $\mathcal{H} = L(l, h, w, y)$ an L-shape related to S. Then

S is symmetric if and only if $wy = 0$ or $(la - yb)(wa - hb) = 0$.

Classification of 3-numerical semigroups by means of L-shapes

Classification

We classify 3-numerical semigroups $S = \langle a, b, N \rangle$ in terms of the number and the degeneration or not of its related L-shapes.

This classification depends on whether $\{a, b, N\}$ is a minimal system of generators for S or not.

As $1 < a < b < N$, $gcd(a, b, N) = 1$, we distinguish two cases:

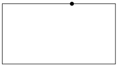
- $\{a, b, N\}$ is not a minimal system of generators.
  In this case $gcd(a, b) \in \{1, a\}$.
- The system $\{a, b, N\}$ is minimal.

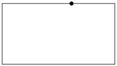
System of generators non minimal

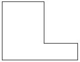
Classification: $gcd(a,b)=1$ and $N \in \langle a,b \rangle$

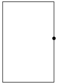
Theorem 1: $S=\langle a,b,N \rangle$, with $1<a<b<N$, $gcd(a,b)=1$, $N=\lambda a+\mu b$ with $\lambda,\mu \in \mathbb{N}\cup\{0\}$. Then

(a.1) $N=\lambda a$, $\lambda \leq b$: $\exists r$, $0 \leq r < \lambda$, $r \equiv -b(\text{mód } \lambda)$,

then there is only one $L(\lambda, a, \lambda - r, 0)$ 

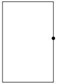
(a.2) $N=\lambda a$, $\lambda > b$: then $\mathcal{H}=L(\lambda,a,b,0)$  and

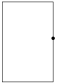
$\mathcal{T}_1(\mathcal{H}) =$  $b \nmid \lambda$

 $b \mid \lambda$

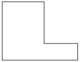
Classification: $\text{mcd}(a,b)=1$ and $N \in \langle a, b\rangle$

(b.1) $N=\mu b$, $\mu \leq a$: $\exists r$, $0 \leq r < \mu$, $r \equiv -a (\text{mód } \mu)$,

then there is only one $L(b, \mu, 0, \mu - r)$ 

(b.2) $N=\mu b$, $\mu > a$: then $\mathcal{H} = L(b, \mu, 0, a)$  and

$\mathcal{T}_2(\mathcal{H}) = \begin{cases} \text{(L-shape)} & a \nmid \mu \\ \text{(rectangle)} & a \mid \mu \end{cases}$

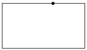
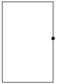


$a \nmid \mu$

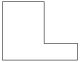
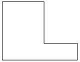
$a \mid \mu$

Classification: $gcd(a,b)=1$ and $N \in \langle a,b \rangle$

(c) $N=\lambda a=\mu b>ab$ if and only if S has related

$\mathcal{H}=L(\lambda,a,b,0)$  and $\mathcal{T}_1(\mathcal{H})=L(b,\mu,0,a)$ 

(d) $N=\lambda a+\mu b$, $a,b\nmid N$: $\Longrightarrow N=\lambda' a+\mu' b$ with $1\le\mu'<a$,

then $\mathcal{H}=L(\lambda'+b,a,b,a-\mu')$  and $\mathcal{T}_1(\mathcal{H})$ 

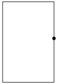
Classification: $gcd(a, b, N) = 1$ and $b = ka$

Theorem 2: $S = \langle a, b, N \rangle$, $1 < a < b < N$, $gcd(a, b, N) = 1$ and $b = ka$. Then

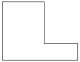
(a) S has related two tiles and at least one has the form

$\mathcal{H} = L(N, 1, k, 0)$ 

(b) S has related two degenerated tiles if and only if $k \mid N$.

This tiles are $\mathcal{H}$ and $\mathcal{T}_1(\mathcal{H}) = L(k, N/k, 0, 1)$ 

(c) If $k \nmid N$, the L-shapes are $\mathcal{H}$ and

$\mathcal{T}_1(\mathcal{H}) = L(k, \lfloor N/k \rfloor + 1, k - r, 1)$  with $N = \lfloor N/k \rfloor k + r$ and $0 < r < k$.

Minimal System of generators

Classification: $gcd(a,b)=1$ y $N \notin \langle a,b\rangle$

N only has values in the set of gaps of the semigroup $S' = \langle a,b\rangle$.

in particular, $b < N < (a-1)(b-1)$.

Theorem 3: $S = \langle a,b,N\rangle$, $1 < a < b < N$, $gcd(a,b)=1$, $N \notin \langle a,b\rangle$:

(a) S is non symmetric if and only if it has related exactly one non degenerated L-shape.

(b) S is symmetric if and only if it has related exactly one degenerated L-shape.

(c) S is not free.

Classification: $gcd(a,b)=p>1$, $a \nmid b$ and $N \notin \langle a,b \rangle$

Lemma: $S=\langle a,b,N\rangle$ minimally generated, $1<a<b<N$ and $gcd(a,b)=p>1$.
Let $S_p=\langle a/p,b/p\rangle$ and $S'=\langle a/p,b/p,N\rangle$.
Then,

(a) S is symmetric if and only if S' is symmetric. (Frőberg, Gotlieb & Hăggkvist, 1987).

(b) S is free if and only if $N \in S_p$.

(c) S is plane curve $N \in S_p$ and $N > \mathrm{mcm}(a,b)$.

Classification: $gcd(a, b) = p > 1$, $a \nmid b$ and $N \notin \langle a, b \rangle$

Theorem 4: $S = \langle a, b, N \rangle$ minimally generated, $1 < a < b < N$ and $gcd(a, b) = p > 1$.
Let $S_p = \langle a/p, b/p \rangle$ and $S' = \langle a/p, b/p, N \rangle$.
Then

(a) S is non symmetric if and only if S' has related exactly one non degenerated L-shape.

(b) S is symmetric, but non free, if and only if S' has related exactly one degenerated L-shape.

(c) S is free if and only if the related L-shapes to S' are ruled by Theorem 1.

(d) S is plane curve if and only if S is free and any L-shape, $L(l, h, w, y)$, related to S' verifies $(l - w)a + (h - y)b > \text{mcm}(a, b)$.

Example: $S_N = \langle 12, 14, N \rangle, \;\; N > 14$ odd; $S'_N = \langle 6, 7, N \rangle, \;\; mcm(12, 14) = 84$

S_N	Condition about S'_N	N	L-shape

Example: $S_N = \langle 12, 14, N\rangle,\ \ N > 14$ odd; $S'_N = \langle 6, 7, N\rangle,\ \ mcm(12, 14) = 84$

S_N	Condition about S'_N	N	L-shape
non symmetric	T4-(a)$\longrightarrow T3-(a)$	$16, 17, 22, 23, 29$	$L(5,4,2,2), L(4,5,3,1), L(6,4,1,2), L(5,5,2,1), L(6,5,2,1)$

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symmetric non free	T4-(b)$\longrightarrow T3-(b)$	15	$L(5,3,1,0)$

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symmetric non free	T4-(b) $\longrightarrow T3-(b)$	15	$L(5,3,1,0)$
free	T4-(c) $\longrightarrow T1-(b{,}1)$	$21, 35$	$L(7,3,0,7)$, $L(7,5,0,1)$
	T4-(c) $\longrightarrow T1-(b{,}2)$	$N = 7\mu$, $\mu = 6p + r$, $0 < r < 6$	$\mathcal{H} = L(7, \mu, 0, 6)$ and $\mathcal{T}_2(\mathcal{H})$
	T4-(c) $\longrightarrow T1-(d)$	$N = 6\lambda + 7\mu$, $\mu = 1, 3, 5$, $\lambda = 7k + r$, $0 < r < 7$	$\mathcal{H} = L(\lambda + 7, 6, 7, 6 - \mu)$ and $\mathcal{T}_1(\mathcal{H})$

Example: $S_N = \langle 12, 14, N \rangle, \;\; N > 14$ odd; $S'_N = \langle 6, 7, N \rangle, \;\; mcm(12,14) = 84$

S_N	Condition about S'_N	N	L-shape

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free non plane curve	free	$15 \leq N \leq 83$, except gaps $17, 23, 29$	
plane curve	free	$N \geq 85$ odd	

Thank you!
for your attention