Constructing the set of complete intersection numerical semigroups with a given Frobenius number

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Given a numerical semigroup Γ, we can associate with Γ its Frobenius number $F(\Gamma)$, and its conductor $c(\Gamma)$. The question we are interested in is the following:
Given an integer c, find the set of semigroups Γ such that $c = c(\Gamma)$.
We shall restrict our study to some special classes of numerical semigroups:

SG of plane curve singularities $\subseteq$ Telescopic SG $\subseteq$ Free SG $\subseteq$ CI SG

The construction has its own interest, but it can also help in the study of some open problems and conjectures (semigroup theory, connected domains).

Example: Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, and let $f(x, y)$ be an irreducible formal power series of $\mathbf{K}[[x, y]]$. The set $\Gamma(f) = \{\mathrm{int}(f, g) | g \in \mathbf{K}[[x, y]]\}$ is a semigroup. This semigroup has a lot of information about f, for example, $c(F) = \mu(f) = \mathrm{int}(f_x, f_y)$ which is the Milnor number of f. Also the structure of the semigroup tells you how to decompose the polar curves of f.

Example: Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, and let $f(x, y)$ be an polynomial of $\mathbf{K}[x, y]$ with one place at infinity. The set $\Gamma(f) = \{\mathrm{int}(f, g) | g \in \mathbf{K}[x, y]\}$ is a semigroup. Again $c(F)$ is the Milnor number of f, and the genus $(1/2)\mu(f))$ is the geometric genus of a smooth fiber $f + \lambda$.

In both cases, define $E(f) = \{F | \Gamma(F) = \Gamma(f)\}$. This is the equivalence class of f. Given $c \in \mathbf{N}$, define E_c to be the union of $E(f), c(F) = c$: what are the generic equations of E_c?

This gives us an algorithmic classification of branches of curves at infinity as well as at finite distance.

Example: Define $\nu(f)$ to be the Turina number (the length of the ideal (f, f_x, f_y)). Locally we have $(1/2)\mu(f) \leq \nu(f) \leq \mu(f)$, the same holds globally provided that f is rational. In both cases, we want to classify curves with $\mu - \nu = i$: $i = 0$ if and only if f is quasi-homogenous -$f = y^a - x^b, \Gamma(f) =< a, b >$. The case $i = 1, 2$ is done locally, the case $i = 1$ is done globally (the main tool is the semigroup of f).

Example: Given two polynomials $p(t) = t^n + a_{n-2}t^{n-2} + \ldots + a_n$, $q(t) = t^m + b_{m-1}t^{m-1} + \ldots + b_m$ where at least one $a_i \neq 0$ and $\gcd(n, m) < \min(m, n)$, give an algebraic proof of the following result: $p(t) - x, q(t) - y$ have at least two ditinct roots for some x, y (using the semigroup of $\mathrm{res}(p(t) - x, q(t) - y)$).

Notations

$\Gamma =< r_0, \ldots, r_h >$

$d_1 = r_0, d_2 = \gcd(d_1, r_1), \ldots, d_{h+1} = 1$

$e_k = \frac{d_k}{d_{k+1}}, k = 1, \ldots, h$

$\Gamma_k =< r_0/d_k, \ldots, r_{k-1}/d_k >, k = 1, \ldots, h+1, c_k = c(\Gamma_k)$.

Ex: $\Gamma = \Gamma_4 =< 8, 12, 26, 51 >$:
$\underline{d} = (8, 4, 2, 1)$
$\underline{e} = (2, 2, 2)$
$\Gamma_3 =< 4, 6, 13 >, \Gamma_2 =< 2, 3 >, \Gamma_1 =< 1 >$
$\underline{c} = (82, 16, 2, 0)$.

The algorithm

The main idea of the algorithm is the following:

Given $c \in \mathbb{N}$

- Find a bound for h
- Find the possible (r_h, d_h, c_{h-1})
- Now we have r_h, the we restart with c_{h-1} and $h-1$, we find this way the possible $(r_{h-1}, d_{h-1}, c_{h-2})$ and so on...

The plane curve singularity case

Let $r_0 < r_1 < \ldots < r_h$. We have $\Gamma = \Gamma(f), f \in \mathbf{K}[[x, y]]$ iff $e_k r_k < r_{k+1}$ and $e_k r_k \in \Gamma_{k-1}$ for all k. In this case,

$$c(\Gamma) = \sum_{i=1}^{h}(e_i - 1)r_i - r_0 + 1 = c(\Gamma_h)d_h + (d_h - 1)(r_h - 1)$$

Proposition: Let Γ be a semigroup associated with an irreducible plane curve singularity, minimally generated by $r_0 < \cdots < r_h$, with $h \geq 2$. Then

- $\frac{1}{3}(5 \cdot 2^{2h-1} - 1) \leq r_h \leq c - \frac{5}{3}2^{2h-2} - 3 \cdot 2^{h-1} + \frac{7}{3}$.
- $c \geq \frac{5}{3}2^{2h} - 3 \cdot 2^h + \frac{4}{3}$
- $h \leq \log_2 \frac{\sqrt{60\,c+1}+9}{10}$

These inequalities give us the set of semigroups with a fixed conductor recursively.

The telescopic case

Take the same notations: Γ is telescopic iff $e_k r_k \in \Gamma_{k-1}$ for all k.

Ex.: $\Gamma =< 4, 6, 9 >$ is Telescopic but it is not the SG of a plane curve singularity.

Proposition Let Γ be a telescopic numerical semigroup minimally generated by $\{r_0 < \cdots < r_h\}$. If $h \geq 2$, then

- $2^{h+1} - 1 \leq r_h \leq c - (h-2)2^h - 1$.
- $c \geq (h-1)2^{h+1} + 2$
- $(h-1)2^{h+1} + 2 \leq c$.

Apply the algorithm as for the previous case.

The free case

The semigroup Γ is free iff $e_k r_k \in \Gamma_{k-1}$ for all k. This notion depends on the order of generators.

Ex: $\Gamma =< 8, 10, 9 >$ is free but not telescopic.

Proposition Let Γ be a free numerical semigroup minimally generated by $\{r_0, \cdots, r_h\}$. If $h \geq 2$, then we have:

- $c = c(\Gamma) = d_h c_{h-1} + (d_h - 1)(r_h - 1)$
- $2^h + 1 \leq r_h \leq c - (h-1)2^{h-1} + 1.$
- $c \geq h2^h$

The complete intersection case

Given A a set positive integers, and $A = A_1 \cup A_2$ a non trivial partition of A, we say that A is the gluing of A_1 and A_2 if $\mathrm{lcm}(d_1, d_2) \in \langle A_1 \rangle \cap \langle A_2 \rangle$, where $d_i = \gcd(A_i)$ and $\langle A_i \rangle$ denotes the monoid generated by A_i, $i = 1, 2$. If A is the minimal system of generators of Γ, and Γ_i is the numerical semigroup generated by A_i/d_i, $i = 1, 2$, we also say that Γ is the gluing of Γ_1 and Γ_2
Let Γ be a numerical semigroup. We know that Γ is a complete intersection if and only if it is the gluing of two complete intersections. We have:

- $c(\Gamma) = d_1 c(\Gamma_1) + d_2 c(\Gamma_2) + (d_1 - 1)(d_2 - 1)$.
- $h \leq \log_2(c(\Gamma)) + 2$

Experimentation

g	cf	t	pc	g	c	f	t	pc	g	c	f	t	pc
0	1	1	1	19	24	24	12	5	38	61	61	37	12
1	1	1	1	20	16	16	11	6	39	100	100	52	16
2	1	1	1	21	27	27	18	9	40	110	109	54	19
3	2	2	2	22	31	31	19	8	41	80	79	47	12
4	3	2	2	23	21	21	13	6	42	122	120	61	20
5	2	2	1	24	36	35	20	11	43	120	120	60	17
6	4	4	3	25	38	38	22	9	44	94	94	48	15
7	5	3	2	26	27	27	16	8	45	143	142	73	22
8	3	2	2	27	46	46	24	11	46	151	149	72	21
9	7	5	4	28	45	45	25	10	47	108	106	57	15
10	8	6	4	29	34	33	20	7	48	158	157	75	24
11	5	4	2	30	57	57	32	13	49	179	179	84	23
12	11	8	5	31	62	62	31	9	50	128	128	68	20
13	11	8	3	32	43	43	25	10	51	197	194	86	26
14	9	7	4	33	65	65	37	14	52	209	207	89	27
15	14	10	6	34	77	76	39	13	53	142	142	76	20
16	17	9	5	35	53	52	29	11	54	229	227	101	30

With $g = 55$, the number of numerical semigroups is 1142140736859, while there are just 2496 symmetric numerical semigroup with genus 55. The average of complete intersections among symmetric numerical semigroups is small, and tiny compared with the whole set of numerical semigroups.

Thank you!!