Numerical semigroups from combinatorial configurations

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Iberian meeting on numerical semigroups - Vila Real 2012

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A (combinatorial) (v, b, r, k)-*configuration* has

- v points,
- b lines,
- r lines through every point and
- k points on every line.

When v and b is not known or important we use the notation (r, k)-*configuration.*

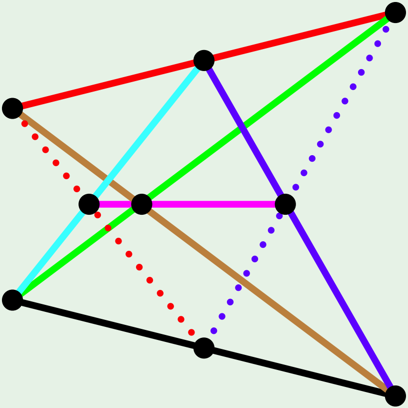
We say that a configuration is **balanced** if $r = k$. This implies $v = b$.

The combinatorial configurations are also

- regular and uniform partial linear spaces,
- a subfamily of the regular and uniform hypergraphs.

Examples of combinatorial configurations

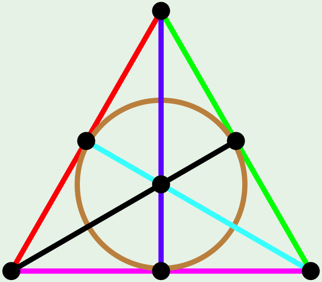
Example (The Pappus configuration)



$$(v, b, r, k) = (9, 9, 3, 3)$$

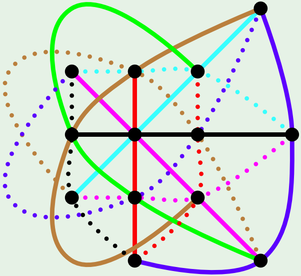
Examples of combinatorial configurations

Example (Finite projective planes)



$\mathbb{P}^2(\mathbb{F}_2)$

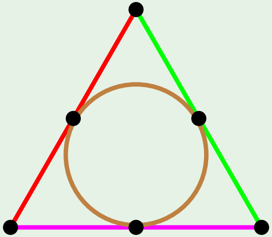
$(v, b, r, k) = (7, 7, 3, 3)$



$\mathbb{P}^2(\mathbb{F}_3)$

$(v, b, r, k) = (13, 13, 4, 4)$

Example (A non-balanced configuration)



$$(v, b, r, k) = (6, 4, 2, 3)$$

Parameter sets of combinatorial configurations

For which parameter sets do combinatorial (v, b, r, k)-configurations exist?

Four parameters are redundant - we only need three:

$$(d, r, k)$$

with

$$v = \frac{dk}{\gcd(r,k)} \text{ and } b = \frac{dr}{\gcd(r,k)}.$$

The set of (r, k)-configurable tuples

Define
$S_{(r,k)} = \left\{ d \in \mathbb{N} \cup \{0\} : \left(\frac{dk}{\gcd(r,k)}, \frac{dr}{\gcd(r,k)}, r, k \right) \text{ is configurable} \right\}.$

Theorem (Bras-Amorós and Stokes)

For every pair of integers $r, k \geqslant 2$, $S_{(r,k)}$ is a numerical semigroup.

Lemma

A set of positive integers generate a numerical semigroup if and only if they are coprime.

It is therefore enough to prove:

- $0 \in S_{(r,k)}$,
- $S_{(r,k)}$ is closed under addition,
- at least two elements of $S_{(r,k)}$ are coprime.

For the first fact, consider the empty configuration.

The two latter facts are proved by combining several configurations into larger configurations.

- Two (r,k)-configurations can be combined so that

  $$(\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1) \oplus (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2) = (\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{I}).$$

  Using the definition of the elements in $S_{(r,k)}$ we get that

  $$d, d' \in S_{(r,k)} \Rightarrow d + d' \in S_{(r,k)}$$

  .

Two coprime elements

- We want to construct two coprime elements in $S_{(r,k)}$.
- We get one element in $S_{(r,k)}$ (say d) associated to the combinatorial configuration obtained by taking parallel classes of a finite affine plane.
- We construct a second combinatorial configuration with an associate integer that is coprime with d.
  Previous work: $md + 1$, with $m = rk/\gcd(r,k)$.
  In this talk we prove $2d - 1$.

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When $r = k$, then there are two constructions (see [Grünbaum]) implying

$$d_1, d_2 \in S_{(r,r)} \Rightarrow d_1 + d_2 - 1$$

and

$$d_1, d_2 \in S_{(r,r)} \Rightarrow d_1 + d_2 + 1.$$

Given an element $d \in S_{(r,r)}$ we get $2d - 1, 2d, 2d + 1 \in S_{(r,r)}$. In particular this is enough for proving finite complement.

Computer search is very hard. Using cyclic configurations, difference sets, Golomb rulers, Thm Bose and Connor we know:

Examples

$r = k$	π	$S_{(r,k)}$	$\setminus\{0\}$					
3	7	7	$\rightarrow$					
4	13	13	$\rightarrow$					
5	21	21	~~22~~	23	$\rightarrow$			
6	31	31	~~32~~	~~33~~	34	$\rightarrow$		
7	43	~~43~~	~~44~~	45	46	47	48	$\rightarrow$	
8	57	57	~~58~~	59	60	61	62	63	$\rightarrow$
9	73	73	~~74~~	75	76	77	78	79	80 $\rightarrow$

For $r = k \in [10, \ldots, 37]$ we know between 35% (for $r = k = 10$) and 89% (for $r = k = 16$) of the small elements of the numerical semigroups $S_{(r,k)}$ [Davydov,Faina,Guilietti,Marcugini,Pambianco].

Configurations with $\gcd(r, k) = 1$

Let $AG(2, q)$ be a finite affine plane over a finite field with q elements. If $q \geqslant \max(r, k)$ then a choice of parallel classes of lines in $AG(2, q)$ gives a combinatorial configuration with associated integer $q \gcd(r, k)$.

Theorem

If $\gcd(r, k) = 1$, *then any prime power* $q \geqslant \max(r, k)$ *belongs to* $S_{(r,k)}$.

Theorem (Bras-Amorós)

Let c be the conductor of a numerical semigroup that contains all prime powers larger than or equal to a given integer n. Then this conductor satisfies

$$c \leqslant 2 \prod_{p \text{ prime},\ p<n} (\lfloor \log_p(n-1) \rfloor + 1),$$

and also

$$c \leqslant \prod_{p \text{ prime},\ p<n} p^{(\lfloor \log_p(n-1) \rfloor)} + 1.$$

Open question: What is the conductor of the numerical semigroup generated by all prime powers larger than n?

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A pattern of length n admitted by a numerical semigroup S is a polynomial $p(X_1, \ldots, X_n)$ with non-zero integer coefficients, such that, for every ordered sequence of n elements $s_1 \geqslant \ldots \geqslant s_n$ from S, we have $p(s_1, s_2, \ldots, s_n) \in S$.

Example

Let S be a numerical semigroup such that for every triple $s_1 \geqslant s_2 \geqslant s_3$ in S we have $s_1 + s_2 - s_3 \in S$. Then the polynomial $X_1 + X_2 - X_3$ is a pattern for S.

A pattern is called linear and homogenous if the pattern polynomial is linear and homogenous.

Theorem

Let $S_{(r,k)}$ be a numerical semigroup associated to the (r,k)-configurations. Then $S_{(r,k)}$ admits the pattern

$$X_1 + X_2 - n$$

for all $n \in [1, \ldots, \gcd(r,k)]$.

Proof.

Take two (r,k)-configurations A and B with associated integers d_A and d_B.

1.
   - Remove $a := nk/\gcd(r,k)$ points $p_1, \ldots, p_a$ on a line L in A.
   - Also remove $b := nr/\gcd(r,k)$ lines $l_1, \ldots, l_b$ through a point p in B.
2.
   - The line L is now missing $nk/\gcd(r,k)$ points.
   - The rest of the $(nk/\gcd(r,k))\,(r-1)$ lines through the removed points are missing one point each.
3.
   - Fill up the missing points on L with p and enough points from l_1.
   - Fill up missing lines through p with enough lines from p.
   - Fill up rest of missing points on lines with points from $l_1, \ldots, l_b$ until through all these points pass r lines.

The result is an (r,k)-configuration with associated integer $d_1 + d_2 - n$. □

Theorem

The conductor c of a numerical semigroup $S_{(r,k)}$ associated to the (r,k)-configurations is bounded by

$$c \leqslant (x+1)m - x\gcd(r,k)$$

where m is the multiplicity of $S_{(r,k)}$ and $x = \left\lfloor \frac{m-2}{\gcd(r,k)} \right\rfloor$.

Proof.

If $d \in S_{(r,k)}$ then $2d - n \in S_{(r,k)}$ for $n \in [1, \gcd(r,k)]$. Therefore the intervals $I_x = [(x+1)d - x\gcd(r,k)), (x+1)d]$ belongs to $S_{(r,k)}$ for $d \in S_{(r,k)}$. If there is a gap between I_x and I_{x-1}, then $(x+1)d - x\gcd(r,k) > xd + 1$, so that $x < \frac{d-1}{\gcd(r,k)}$. Hence, for $x \geqslant \left\lfloor \frac{d-2}{\gcd(r,k)} \right\rfloor$ there are no gaps between the intervals I_x. □

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Admissible numerical semigroups $S_{(r,k)}$ for a fixed integer d

Lemma

Suppose that there exists a configuration with parameters (v, b, r, k)*. Then we have*

$$v \geqslant r(k-1)+1$$

and also

$$b \geqslant k(r-1)+1.$$

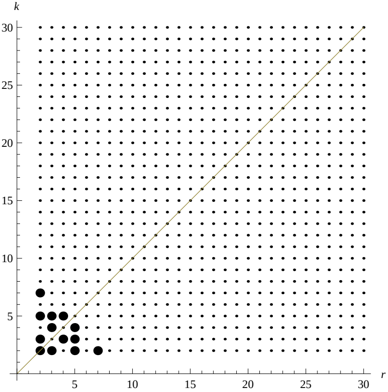
In the following, remember that $v = \frac{dk}{\gcd(r,k)}$ and $b = \frac{dr}{\gcd(r,k)}$.

Definition

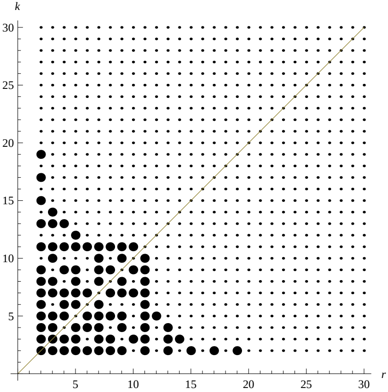
For $d \in \mathbb{N}$ we denote

$$R_d = \{(r,k) \in \mathbb{N}^2_{\geqslant 2} : r(k-1)+1 \leqslant \frac{dk}{\gcd(r,k)} \text{ and } k(r-1)+1 \leqslant \frac{dr}{\gcd(r,k)} \}.$$

Examples



R_4



R_{10}

Properties of the set R_d

Proposition

1. *The set R_d is finite.*
2. *The set R_d is symmetric, in the sense that if $(r,k) \in R_d$ then $(k,r) \in R_d$.*
3. *If $d < d'$ then $R_d \subseteq R_{d'}$.*
4. *The l_1 norm of a point $P = (r,k) \in R_d$ satisfies $l_1(P) = |r| + |k| = r + k \leqslant 2d + 2$ and when $r \neq k$ we have $l_1(P) = |r| + |k| = r + k \leqslant 2d + 1$.*

Corollary

The parameters r and k of a combinatorial (r,k)-configuration with associated integer d satisfy

$$r + k \leqslant 2d + 1.$$

Observe that the fact that $(r,k) \in R_d$ does not imply that $S_{(r,k)}$ actually contains d. For example

$$R_{43} = \{(r,k) \in \mathbb{N}^2 : \quad r(k-1)+1 \geqslant \tfrac{43k}{\gcd(r,k)}, \quad k(r-1)+1 \geqslant \tfrac{43r}{\gcd(r,k)} \text{ and } r,k \geqslant 2\},$$

which means that $(7,7) \in R_{43}$, but if 43 was in $S_{(7,7)}$, then there would be a $(43,43,7,7)$-configuration and this configuration would be a finite projective plane of order 6. But there is no finite projective plane of order 6, so 43 can not be in $S_{(7,7)}$.

Numerical semigroups $S_{(r,k)}$ containing an certain integer

Difference of regions R_d of parameters (r,k) admitting d	Numerical semigroups $S_{(r,k)}$
$R_2 = \{(2,3),(3,2)\}$	$S_{(r,k)} = \{0, 2 \rightarrow\}$
$R_3 \setminus R_2 = \{(2,2),(2,5),(5,2),(3,4),(4,3)\}$	$S_{(r,k)} = \{0, 3 \rightarrow\}$
$R_4 \setminus R_3 = \{(2,7),(7,2),(3,5),(5,3),(4,5),(5,4)\}$	$S_{(r,k)} = \{0, 4 \rightarrow\}$
$R_5 \setminus R_4 = \{(2,4),(4,2),(2,9),(9,2),(3,7),(7,3),(5,6)(6,5)\}$	$S_{(r,k)} = \{0, 5 \rightarrow\}$ except possibly $S_{(5,6)} = S_{(6,5)} = \{0, 5, 6, 7 \rightarrow\}$

- **Does all numerical semigroups appear as the numerical semigroup of a the (r, k)-configurations for some parameter set (r, k)?**

  No! For example, a non-ordinary numerical semigroup with multiplicity 2 is NOT a numerical semigroup of the (r, k)-configurations for any (r, k).

- **Can the same numerical semigroup appear as attached to the (r, k)-configurations several parameters (r, k)?**

  Yes! For example, the numerical semigroups $S_{(5,2)} = S_{(2,5)} = S_{(2,2)} = \{0, 3 \rightarrow\}$.

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- A.A. Davydov, G. Faina, M. Giulietti, S. Marcugini, F. Pambianco *On constructions and parameters of symmetric configurations* v_k, Arxiv:1203.0709v1.
- B. Grünbaum, "Configurations of Points and Lines." American Mathematical Society, Providence, RI, 2009.
- K. Stokes and M. Bras-Amorós (2011) *Associating a numerical semigroup to the triangle-free configurations*, Advances in Mathematics of Communications, 5:2, pp. 351-371.

Thank you very much!