Non-homogeneous patterns on numerical semigroups

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Reed-Solomon codes

- Given n pairwise distinct elements $\alpha_1, \ldots, \alpha_n$ of a finite field $\mathbb{F}_q$, the Reed-Solomon code $RS_{\alpha_1,\ldots,\alpha_n}(k)$ is defined by:

$$\{(f(\alpha_1), \ldots, f(\alpha_n)) : f \in \mathbb{F}_q[x], \deg(f) < k\}.$$

- Then, the length of $RS_{\alpha_1,\ldots,\alpha_n}(k)$ is n and it is bounded by the field size q.

Reed-Muller codes

- A generalization of this is that of Reed-Muller codes. That is, given n pairwise distinct affine points $P_1, \ldots, P_n$ of the affine space $\mathbb{F}_q^m$, the Reed-Muller code $RM_{P_1,\ldots,P_n}(k)$ is definied by:

$$\{(f(P_1), \ldots, f(P_n)) : f \in \mathbb{F}_q[x_1, \ldots, x_m], \deg(f) < k\}.$$

- Then, the length of $RM_{P_1,\ldots,P_n}(k)$ is n and it is bounded by the size of $\mathbb{F}_q^m$, that is, q^m.

Algebraic codes

- Algebraic geometry codes generalize this giving codes attaining very important asymptotic bounds. Given n pairwise distinct places $P_1, \ldots, P_n$ of degree one of an algebraic function field $F/\mathbb{F}_q$, and a divisor G with support not included in $\{P_1, \ldots, P_n\}$, the geometric Goppa code $C_{P_1,\ldots,P_n}(G)$ is definied by

$$\{(f(P_1), \ldots, f(P_n)) : f \in L(G)\}.$$

- Then, the length of $C_{P_1,\ldots,P_n}(G)$ is n and it is bounded by the number of places of degree one of $F/\mathbb{F}_q$.

Number of points and codes

- Thus, an important problem of algebraic coding theory is bounding the number of points of an algebraic curve.

Weierstrass semigroup

Given an algebraic curve, the Weierstrass semigroup Λ of a rational point, is the set of pole orders of rational functions having only poles in that point.

It satisfies:

- $\Lambda \subseteq \mathbb{N}_0$.
- $0 \in \Lambda$.
- if $a, b \in \Lambda$ then $a + b \in \Lambda$.
- $\#\mathbb{N}_0 \setminus \Lambda =$ genus of the curve.

Hence, the Weierstrass semigroup contains information on the genus.

Weierstrass semigroup

- The generators of Λ are those non-zero elements that can not be expressed as a sum of two other elements in Λ.
- If the generators of Λ are $\lambda_1, \ldots, \lambda_n$ then $\Lambda = \{a_1\lambda_1 + \ldots + a_n\lambda_n : a_i, \ldots, a_n \in \mathbb{N}_0\}$.
- We denote this by $\Lambda = \langle \lambda_1, \ldots, \lambda_n \rangle$.

Existing bounds

- Depending on the genus of the curve:
  - Serre-Hasse-Weil bound
    Let C/F_q be a curve of genus g then the number of $F_q - rational$ points satisfies the following inequality:
    $\#N_q(g) \leq q + 1 + g\left[2\sqrt{q}\right]$

Existing bounds

- Depending on Weierstrass semigroups:
  1. Geil-Matsumoto:
     $N_q(\Lambda) \leq GM_q(\Lambda) = \#(\Lambda \setminus \cup_{\lambda_i \text{ generator of } \Lambda}(q\lambda_i + \Lambda)) + 1$
  - Pros and cons:
    - $+$ Best known bound related to Weierstrass semigroups (for some values it is better than Serre-Hasse-Weil bound).
    - - not simple.

Example with $q = 3$ and $\Lambda = \langle 5, 7 \rangle$

$$GM_q(\Lambda) = \#(\Lambda \setminus \cup_{\lambda_i \text{ generator of } \Lambda}(q\lambda_i + \Lambda)) + 1$$

- Λ:
  - 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 ...
- $q5 + \Lambda$:
  - 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 ...
- $q7 + \Lambda$:
  - 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 ...
- $\Lambda \setminus \{(q5 + \Lambda) \cup (q7 + \Lambda)\}$:
  - 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 ...
- $GM_q(\Lambda) = \# \{0,5,7,10,12,14,17,19,24\} + 1 = 10$

Existing bounds

- Depending on Weierstrass semigroups:

  2. Lewittes:
     $N_q(\Lambda) \leq L_q(\Lambda) = q\lambda_1 + 1$

  - Pros and cons:
    - - Weaker than Geil-Matsumoto.
    - + simpler.

Example with $q = 3$ and $\Lambda = \langle 5, 7 \rangle$

$N_q(\Lambda) \leq L_q(\Lambda) = q\lambda_1 + 1$

- $L_q(\Lambda) = 3 \cdot 5 + 1 = 16$

A lot more faster!!

A closed formula for GM bound for semigroups with two generators

Lemma

The Geil-Matsumoto bound for the semigroup generated by a *and* b *with* $a < b$ *is:*

$$GM_q(\langle a, b\rangle) = 1 + \sum_{n=0}^{a-1} \min\left(q, \left\lceil \frac{q-n}{a} \right\rceil \cdot b\right) =$$

$$\begin{cases} 1 + qa & \text{if } q \leqslant \lfloor \frac{q}{a} \rfloor b \\ 1 + (q \bmod a)q + (a - (q \bmod a))\lfloor \frac{q}{a} \rfloor b & \text{if } \lfloor \frac{q}{a} \rfloor b < q \leqslant \lceil \frac{q}{a} \rceil b \\ 1 + ab\lceil \frac{q}{a} \rceil - (a - (q \bmod a))b & \text{if } q > \lceil \frac{q}{a} \rceil b \end{cases}$$

Coincidences of $GM(\Lambda) = L(\Lambda)$

- We proved that:
  $GM_q(\langle a, b\rangle) = L_q(\langle a, b\rangle)$ if and only if $q \leqslant \lfloor \frac{q}{a} \rfloor b$.
- Otherwise the Geil-Matsumoto bound always gives an improvement with respect to the Lewittes's bound.
- We would wish to generalize this to semigroups with any number of generators.

Coincidences of $GM(\Lambda) = L(\Lambda)$

Lemma

It holds

$$GM_q(\langle\lambda_1, \lambda_2, \ldots, \lambda_n\rangle) = L_q(\langle\lambda_1, \lambda_2, \ldots, \lambda_n\rangle) = q\lambda_1 + 1$$

if and only if $q(\lambda_i - \lambda_1) \in \Lambda$ *for all* i *with* $2 \leq i \leq n$

Analysis of lemmas

	Lewittes = Geil-Matsumoto				
Genus	q=2	q=3	q=9	q=16	q=256
2	50.00%	100%	100%	100%	100%
3	25.00%	75.00%	100%	100%	100%
4	42.86%	57.14%	100%	100%	100%
5	33.33%	41.67%	91.67%	100%	100%
6	21.74%	43.48%	86.96%	100%	100%
7	17.95%	41.03%	87.18%	100%	100%
8	14.93%	37.31%	85.07%	100%	100%
9	11.02%	33.05%	88.14%	98.31%	100%
10	8.82%	29.90%	88.24%	95.59%	100%
11	7.58%	25.95%	84.55%	92.71%	100%
12	6.59%	23.48%	78.89%	90.88%	100%
13	5.69%	21.48%	73.73%	89.81%	100%
14	5.02%	18.90%	69.76%	88.66%	100%
15	4.10%	16.63%	66.26%	87.68%	100%
16	3.45%	14.77%	63.23%	87.22%	100%
17	2.92%	13.10%	60.66%	87.00%	100%
18	2.38%	11.66%	58.74%	87.03%	100%
19	1.93%	10.40%	57.06%	86.71%	100%
20	1.60%	9.28%	55.71%	85.43%	100%
21	1.31%	8.34%	54.67%	83.03%	100%
22	1.09%	7.48%	53.95%	80.14%	100%
23	0.90%	6.70%	53.29%	77.41%	100%
24	0.75%	6.02%	52.46%	75.16%	100%
25	0.63%	5.42%	51.33%	73.37%	100%
26	0.53%	4.90%	49.94%	71.94%	100%
27	0.45%	4.45%	48.39%	70.75%	100%
28	0.38%	4.07%	46.81%	69.73%	100%
29	0.32%	3.74%	45.25%	68.76%	100%
30	0.27%	3.44%	43.76%	67.80%	100%

Table: Portion of semigroups for which the Lewittes and the Geil-Matsumoto bounds coincide.

Motivation behind patterns

- Arf semigroups appear in many theoretical problems in algebraic geometry as well as in some applied areas such as coding theory.
- Arf semigroups are those semigroups such that for any elements x_1, x_2, x_3 in the semigroup with $x_1 \geq x_2 \geq x_3$, the integer $x_1 + x_2 - x_3$ also belongs to the semigroup.
- This definition inspired studying the so-called patterns on numerical semigroups.

Patterns on numerical semigroups

- Patterns on numerical semigroups are multivariate polynomials such that evaluated at any decreasing sequence of elements of the semigroup give integers belonging to the semigroup.
- For their simplicity, and for their inspiration in Arf semigroups, patterns were first defined to be linear and homogeneous.
- However, other families of numerical semigroups have appeared lately in very different areas of applied mathematics which satisfy linear non-homogeneous patterns.

Homogeneous patterns

- Homogeneous patterns were first introduced and studied by Maria Bras-Amorós and Pedro A. García-Sánchez in Patterns on numerical semigroups published in Linear Algebra Appl., 414(2-3):652-669, 2006.
- In that paper a semigroup was said to *admit* a linear homogeneous pattern $p(x_1,\dots,x_n)=\sum_{i=1}^n a_i x_i$ if for every n elements $s_1,\dots,s_n$ in Λ with $s_1\geq s_2\geq\dots\geq s_n$ the integer $p(s_1,\dots,s_n)$ belongs to Λ.

Non-homogeneous patterns

- A non-homogeneous pattern can be expressed in the following form: $\sum_{i=1}^{n} a_i x_i + a_0$.
- We will say that a numerical semigroup $admits$ a non-homogeneous pattern $p(x_1, \ldots, x_n)$ if for every n *non-zero* elements $s_1, \ldots, s_n$ in Λ with $s_1 \geq s_2 \geq \cdots \geq s_n$ the integer $p(s_1, \ldots, s_n)$ belongs to Λ.

Geil-Matsumoto pattern

- We proved that the Geil-Matsumoto bound and the Lewittes' bound coincide if and only if $qx - qm \in \Lambda$ for all $x \in \Lambda$, where m is the multiplicity of Λ.
- This example, for a fixed q (field size) and a fixed m (multiplicity), is related to the non-homogeneous pattern $qx_1 - qm$.

MED-semigroups pattern

- The number of generators (usually referred to as the embedding dimension) is bounded by the multiplicity and those numerical semigroups for which the number of generators equals the multiplicity are said to be of maximal embedding dimension (MED).

MED-semigroups pattern

- MED-semigroups are characterized by the fact that for any two non-zero elements x, y of the semigroup one has that $x+y-m$ belongs to the semigroup where m is its multiplicity.
- This example, for a fixed m (multiplicity) is related to the non-homogeneous pattern x_1+x_2-m.

Combinatorial configurations pattern

- A (v, b, r, k)-combinatorial configuration is an incidence structre with a set of v points and a set of b lines such that each line contains k points, each point is contained in r lines, and any two distinct lines are incident with at most one point or, equivalently, any two distinct points coincide in at most one line.
- It is easy to prove that if a (v, b, r, k)-configuration exists then necessarily $vr = bk$ and so, there exists an integer d such that $(v, b, r, k) = (d\frac{k}{gdc(r,k)}, d\frac{k}{gcd(r,k)}, r, k)$.

Combinatorial configurations pattern

- For a fixed pair r, k, the set $D_{r,k}$ of all integers d such that there exists a $(d\frac{k}{gdc(r,k)}, d\frac{k}{gcd(r,k)}, r, k)$-configuration, is a numerical semigroup (*M. Bras-Amorós, K. Stokes: The semigroup of combinatorial configurations, Semigroup Forum, 1(84):91-96, 2012*).
- For all integers $d, d' \in D_{r,k}$ Klara Stokes proved that $d + d' - 1 \in D_{r,k}$.
- This example is related to the non-homogeneous pattern $x_1 + x_2 - 1$.

Homogeneous admissibe patterns

We denote by $S(p)$ the set of all numerical semigroups admitting p. For the case of homogeneous patterns it was proved that the following conditions are equivalent for a pattern $\sum_{i=1}^{n} a_i x_i$.

(a) $S(p) \neq 0$.

(b) $\mathbb{N}_0 \in S(p)$

(c) $\sum_{i=1}^{j} a_i x_i \geq 0$ for all $j \leq n$.

Non-homogeneous admissible patterns

We will prove an equivalent result for non-homogeneous patterns.

- When dealing with non-homogeneous patterns, the role that $\mathbb{N}_0$ played for homogeneus patterns will be played by an ordinary semigroup.
- That is, a semigroup of the form $\{0\} \cup (m + \mathbb{N}_0)$ for some integer m.
- This semigroup is represented by $\{0, m, \rightarrow\}$.

Non-homogeneous admissible patterns

Lemma

If the non-homogeneous pattern $p = \sum_{i=1}^{n} a_i x_i + a_0$ *is admitted at least by one semigroup* $(S(p) \neq 0)$ *then* $\sum_{i=1}^{j} a_i \geq 0$ *for all* $j \leq n$.

Lemma

Suppose that the non-homogeneous pattern $p = \sum_{i=1}^{m} a_i x_i + a_0$ *satisfies* $\sigma_j = \sum_{i=1}^{j} a_i \geq 0$ *for all* $j \leq n$. *Let* m *be any positive integer satisfying* $m \geq -\frac{a_0}{\sigma_n - 1}$. *Then the ordinary semigroup* $\{0, m, \rightarrow\}$ *admits* p.

Non-homogeneous admissible patterns

Theorem

The next conditions are equivalent for a pattern $p = \sum_{i=1}^{n} a_i x_i + a_0$ *where* $\sigma_j = \sum_{i=1}^{j} a_i$

(a) $S(p) \neq 0$.

(b) $\{0, m, \rightarrow\} \in S(p)$ *for all* $m \geq -\frac{a_0}{\sigma_n - 1}$.

(c) $\sigma_j \geq 0$ *for all* $j \leq n$.

End

Thanks for listening.