Two-extension of a numerical semigroup with embedding dimension two

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- Introduction
- The characterization of $\mathcal{F}(n_1, n_2)$
- The genus and the Frobenius number
- The number of 2-extensions

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- Cardinality of its minimal system of generators is the embedding dimension of S, denoted by $\mathrm{e}(S) = p$.
- The elements of $\mathbb{N} \setminus S$ are called gaps of S and its cardinality is the genus of S, denoted by $\mathrm{g}(S)$.
- The greatest integer not in S is the Frobenius number, denoted by $\mathrm{F}(S)$.

Introduction

$\bullet$Let S and $\overline{S}$ be two numerical semigroups such that $S \subseteq \overline{S}$ and let $d \in \mathbb{N}$. We say that $\overline{S}$ is a d-extension of S if $d\overline{S} = \{dx \ : \ x \in \overline{S}\} \subset S$.

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• For $n_1, n_2 \in \mathbb{N}$ greater or equal than two and gcd $\{n_1, n_2\} = 1$ we denote by $\mathcal{F}(n_1, n_2) = \{S \ : \ 2S \subset \langle n_1, n_2 \rangle\}$, the set of numerical semigroups S such that S is a 2-extension of $\langle n_1, n_2 \rangle$.

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• We characterize the elements of $\mathcal{F}(n_1, n_2)$ and we give formulas for $\mathrm{F}(S)$ and $\mathrm{g}(S)$ for the numerical semigroups in this set.

•We show how we compute all numerical semigroups S in $\mathcal{F}(n_1, n_2)$ with given $\mathrm{e}(S)$ and we obtain formulas $\#\mathcal{F}(n_1, n_2)$.

The characterization of $\mathcal{F}(n_1, n_2)$

- We denote by $D(n_1, n_2) = \{x \notin \langle n_1, n_2 \rangle \ : \ 2x \in \langle n_1, n_2 \rangle\}$.

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Lemma

Let $x \in \mathbb{N}$. Then $x \in D(n_1, n_2)$ if and only if $x = n_1 n_2 - an_1 - bn_2$ for some nonnegative integers a and b, such that $1 \leq a \leq \frac{n_2}{2}$ and $1 \leq b \leq \frac{n_1}{2}$.

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Proposition

Given a set X of integers, we have that
$\mathcal{F}(n_1, n_2) = \{(X \cup \{0\}) + \langle n_1, n_2 \rangle \ : \ X \subseteq D(n_1, n_2)\}$.

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- We define an order relation, $\leq_D$, in $D(n_1, n_2)$ as follows: $x \leq_D y$ if $y - x \in \langle n_1, n_2 \rangle$.

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Theorem

we have that $\mathcal{F}(n_1, n_2) = \{(X \cup \{0\}) + \langle n_1, n_2 \rangle \ : \ X \text{ is a subset of incomparable elements of } D(n_1, n_2)\}$. *Moreover, let* X *and* Y *be two subsets of incomparable elements of* $D(n_1, n_2)$. *Then* $(X \cup \{0\}) + \langle n_1, n_2 \rangle = (Y \cup \{0\}) + \langle n_1, n_2 \rangle$ *if and only if* $X = Y$.

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• We define in $\mathcal{B}(n_1, n_2)$ the following order relation: $(a, b) \leq (a', b')$ if $(a' - a, b' - b) \in \mathbb{N} \times \mathbb{N}$

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The correspondence $\theta : \mathcal{B}(n_1, n_2) \longrightarrow D(n_1, n_2)$, defined by $\theta(a, b) = n_1 n_2 - an_1 - bn_2$, is a bijective map. Moreover $(a, b) \leq (a', b')$ if and only if $\theta(a', b') \leq_D \theta(a, b)$.

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Corollary

The cardinality of set $D(n_1, n_2)$ is equal to $\lfloor \frac{n_2}{2} \rfloor \lfloor \frac{n_1}{2} \rfloor$.

The characterization of $\mathcal{F}(n_1, n_2)$

We define $\theta_* : P\big(\mathcal{B}(n_1, n_2)\big) \longrightarrow P\big(D(n_1, n_2)\big)$, by $\theta_*(A) = \{\theta(a) \; : \; a \in A\}$.
This map θ_* is called the direct image of θ.

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We have that $\mathcal{F}(n_1, n_2) =$
$\big\{\big(\theta_*(A) \cup \{0\}\big) + \langle n_1, n_2\rangle \ : \ A \text{ is a subset of incomparable elements of } \mathcal{B}(n_1, n_2)\big\}$
Moreover, let A *and* A' *be two subsets of incomparable elements of* $\mathcal{B}(n_1, n_2)$*.*
Then $\big(\theta_*(A) \cup \{0\}\big) + \langle n_1, n_2\rangle = \big(\theta_*(A') \cup \{0\}\big) + \langle n_1, n_2\rangle$ *if and only if* $A = A'$*.*

The genus and the Frobenius number

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> The number of 2-extensions

The next result is deduced from (The Frobenius problem for numerical semigroups)

Proposition

Let $A = \{(a_1, b_1) \dots, (a_p, b_p)\}$ *be a subset of incomparable elements of* $\mathcal{B}(n_1, n_2)$ *such that* $a_1 > a_2 > \cdots > a_p$, $b_1 < b_2 < \cdots < b_p$ *and* $S = \big(\theta_*(A) \cup \{0\}\big) + \langle n_1, n_2 \rangle$. *Then:*

1. $\mathrm{g}(S) = \frac{(n_1-1)(n_2-1)}{2} - \big(a_1 b_1 + a_2(b_2 - b_1) + \ldots + a_p(b_p - b_{p-1})\big)$;
2. $\mathrm{F}(S) = \max\{\theta(a_1+1, 1), \theta(a_2+1, b_1+1), \dots, \theta(a_p+1, b_{p-1}+1), \theta(1, b_p+1)\}$.

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Let $A = \{(a_1, b_1) \dots, (a_p, b_p)\}$ be a subset of incomparable elements of $\mathcal{B}(n_1, n_2)$ such that $a_1 > a_2 > \cdots > a_p$, $b_1 < b_2 < \cdots < b_p$ and $S = \big(\theta_*(A) \cup \{0\}\big) + \langle n_1, n_2 \rangle$. Then:

1. $\mathrm{g}(S) = \frac{(n_1-1)(n_2-1)}{2} - \big(a_1 b_1 + a_2(b_2 - b_1) + \ldots + a_p(b_p - b_{p-1})\big)$;
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We give an example that illustrates this procedure.

Example

Take $n_1 = 9$ and $n_2 = 11$. Then $\mathcal{B}(9, 11) = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\}$. It is clear that $A = \{(4, 1), (3, 3), (2, 4)\}$ is a subset of incomparable elements of $\mathcal{B}(9, 11)$. We have that $S = \{0, \theta(4, 1), \theta(3, 3), \theta(2, 4)\} + \langle 9, 11 \rangle$ is a numerical semigroup in $\mathcal{F}(9, 11)$. We get $\mathrm{g}(S) = \frac{8 \cdot 10}{2} - (4 + 6 + 2) = 40 - 12 = 28$ and $\mathrm{F}(S) = \max\{\theta(5, 1), \theta(4, 2), \theta(3, 4), \theta(1, 5)\} = \max\{43, 41, 28, 35\} = 43$.

The number of 2-extensions

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Let $(X \cup \{0\}) + \langle n_1, n_2 \rangle$ be a numerical semigroup with $X \subset D(n_1, n_2)$, next result gives us $\mathrm{e}(X \cup \{0\}) + \langle n_1, n_2 \rangle$.

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Proposition

Suppose that $\min\{n_1, n_2\} \geq 3$. *Let* X *be subsets of incomparable elements of* $D(n_1, n_2)$ *and let* $S = (X \cup \{0\}) + \langle n_1, n_2 \rangle$. *Then:*

1. *if* n_1 *and* n_2 *are odd, then* $X \cup \{n_1, n_2\}$ *is a minimal system of generators of* S*;*
2. *if* n_1 *is odd,* n_2 *even and* $\frac{n_2}{2} \notin X$ *, then* $X \cup \{n_1, n_2\}$ *is a minimal system of generators of* S*;*
3. *if* n_1 *is odd,* n_2 *even and* $\frac{n_2}{2} \in X$*, then* $X = \left\{\frac{n_2}{2}\right\}$ *and* $\left\{n_1, \frac{n_2}{2}\right\}$ *is a minimal system of generators of* S*.*

The number of 2-extensions

Given a positive integer n, we denote by
$\mathcal{F}^n(n_1, n_2) = \{S \in \mathcal{F}(n_1, n_2) \ : \ \mathrm{e}(S) = n\}$

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$\mathcal{F}^n(n_1, n_2) = \{S \in \mathcal{F}(n_1, n_2) \ : \ \mathrm{e}(S) = n\}$

Corollary

Let $\min\{n_1, n_2\} \geq 3$*. Then:*

1. *if* $S \in \mathcal{F}(n_1, n_2)$*, then* $\mathrm{e}(S) \geq 2$*;*
2. *if* n_1 *and* n_2 *are odd, then* $\mathcal{F}^2(n_1, n_2) = \{\langle n_1, n_2\rangle\}$*;*
3. *if* n_1 *is odd and* n_2 *even, then* $\mathcal{F}^2(n_1, n_2) = \left\{\langle n_1, n_2\rangle, \left\langle n_1, \frac{n_2}{2}\right\rangle\right\}$*;*
4. *if* n_1 *and* n_2 *are odd, then* $\mathcal{F}^3(n_1, n_2) = \{\langle n_1, n_2, x\rangle \ : \ x \in D(n_1, n_2)\}$ *and* $\#\mathcal{F}^3(n_1, n_2) = \lfloor \frac{n_1}{2} \rfloor \lfloor \frac{n_2}{2} \rfloor$*;*
5. *if* n_1 *is odd and* n_2 *even, then* $\mathcal{F}^3(n_1, n_2) = \left\{\langle n_1, n_2, x\rangle \ : \ x \in D(n_1, n_2) \setminus \left\{\frac{n_2}{2}\right\}\right\}$ *and* $\#\mathcal{F}^3(n_1, n_2) = \lfloor \frac{n_1}{2} \rfloor \lfloor \frac{n_2}{2} \rfloor - 1$.

The number of 2-extensions

Our next goal is to find the cardinality of the set $\mathcal{F}^n(n_1, n_2)$ with $n \geq 4$.

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Theorem

Let $\min\{n_1, n_2\} \geq 3$. *Then:*

1. *if* $S \in \mathcal{F}(n_1, n_2)$, *then* $\mathrm{e}(S) \leq \min\left\{\lfloor \frac{n_1}{2} \rfloor, \lfloor \frac{n_2}{2} \rfloor\right\} + 2$;
2. *if* $m \in \left\{2, \dots, \min\left\{\lfloor \frac{n_1}{2} \rfloor, \lfloor \frac{n_2}{2} \rfloor\right\}\right\}$, $A = \{a_1 > a_2 > \cdots > a_m\} \subseteq \left\{1, \dots, \lfloor \frac{n_2}{2} \rfloor\right\}$ *and* $B = \{b_1 < b_2 \cdots < b_m\} \subseteq \left\{1, \dots, \lfloor \frac{n_1}{2} \rfloor\right\}$, *then* $\{0, \theta(a_1, b_1), \theta(a_2, b_2), \dots, \theta(a_m, b_m)\} + \langle n_1, n_2 \rangle$ *is an element in* $\mathcal{F}^{m+2}(n_1, n_2)$. *Furthermore every element in* $\mathcal{F}^{m+2}(n_1, n_2)$ *is of this form;*
3. *if* $n \in \left\{4, \dots, \min\left\{\lfloor \frac{n_1}{2} \rfloor, \lfloor \frac{n_2}{2} \rfloor\right\} + 2\right\}$, *then* $\#\mathcal{F}^n(n_1, n_2) = \binom{\lfloor \frac{n_1}{2} \rfloor}{n-2}\binom{\lfloor \frac{n_2}{2} \rfloor}{n-2}$.

The number of 2-extensions

Corollary

If $\min\{n_1, n_2\} \geq 3$*, then*

$$\#\mathcal{F}(n_1, n_2) = \sum_{i=0}^{\min\{\lfloor \frac{n_1}{2} \rfloor, \lfloor \frac{n_2}{2} \rfloor\}} \binom{\lfloor \frac{n_1}{2} \rfloor}{i} \binom{\lfloor \frac{n_2}{2} \rfloor}{i}.$$

The number of 2-extensions

Example

Assume that $n_1 = 9$ and $n_2 = 11$. We compute the elements of $\mathcal{F}^5(9, 11)$.
From previous Theorem, we know that $\#\mathcal{F}^5(9, 11) = \binom{4}{3}\binom{5}{3} = 40$.
As consequence the previous results, we obtain the elements of $\mathcal{F}^5(9, 11)$, in the following way: choose A and B subsets of $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4\}$, respectively, with cardinality 3. Assume that $A = \{2, 3, 4\}$ and $B = \{1, 3, 4\}$. Then $a_1 = 4 > a_2 = 3 > a_1 = 2$ and $b_1 = 1 < b_2 = 3 < b_3 = 4$. From this we have the set $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} = \{(4, 1), (3, 3), (2, 4)\}$ that is a subset of incomparable elements of $\mathcal{B}(9, 11)$. Then
$\{0, \theta(4, 1), \theta(3, 3), \theta(2, 4)\} + \langle 9, 11\rangle = \langle 9, 11, \theta(4, 1), \theta(3, 3), \theta(2, 4)\rangle =$
$\langle 9, 11, 37, 39, 52\rangle$ is a numerical semigroup in $\mathcal{F}^5(9, 11)$.
In order to conclude, we obtain $\#\mathcal{F}(9, 11) = \sum_{i=0}^{4} \binom{4}{i}\binom{5}{i} =$
$\binom{4}{0}\binom{5}{0} + \binom{4}{1}\binom{5}{1} + \binom{4}{2}\binom{5}{2} + \binom{4}{3}\binom{5}{3} + \binom{4}{4}\binom{5}{4} = 1 + 20 + 60 + 40 + 5 = 126.$
Moreover, we obtain that $\#\mathcal{F}^2(9, 11) = 1$, $\#\mathcal{F}^3(9, 11) = 20$, $\#\mathcal{F}^4(9, 11) = 60$, $\#\mathcal{F}^5(9, 11) = 40$ and $\#\mathcal{F}^6(9, 11) = 5$.

Main references

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