# Delta Sets of Numerical Monoids Using Non-Minimal Generators

Scott Chapman

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July 20, 2012

# This Talk is Based on the Paper:

[1] S. T. Chapman, G. Daigle, R. Hoyer and N. Kaplan, "Delta sets of numerical monoids using non-minimal sets of generators," *Comm. Algebra.* **38**(2010), 2622–2634.

# Other papers related to this talk:


[2] S. T. Chapman, C. Bowles, N. Kaplan and D. Reiser, "On Delta sets of numerical monoids," *Journal Algebra Appl.* **5**(2006), 1–24.

[3] J. Amos, S. T. Chapman, N. Hine and J. Paixao, "Sets of lengths do not characterize numerical monoids," *Integers* **7**(2007), #A50.

[4] S. T. Chapman, R. Hoyer and N. Kaplan, "Delta sets of numerical monoids are eventually periodic," *Aequationes Math* **77**(2009), 273–279.

[5] S. T. Chapman, N. Kaplan, T. Lemburg, A. Niles and C. Zlogar, "Shifts of generators and delta sets of numerical monoids," to appear in *J. Comm. Algebra*.

[6] S. T. Chapman, P.A. García-Sánchez, D. Llena, A. Malyshev and D. Steinberg, "On the Delta sets and minimal presentations of a finitely generated commutative monoid," *Arabian J. Math.* **1**(2012), 53–61.

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# Definitions

Let $M$ be a commutative cancellative atomic monoid.

Let $\mathcal{I}(M)$ be the set of irreducible elements of $M$

and $M^{\times}$ its set of units.

For $m \in M \backslash M^{\times}$, set

$$\mathcal{L}(m) = \{ \, t \in \mathbb{N} \mid \exists \, x_1, \ldots, x_t \in \mathcal{I}(M) \text{ with } m = x_1 \cdots x_t \, \}.$$

The set $\mathcal{L}(m)$ is called the *set of lengths* of $m$.

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If $m \in M \backslash M^{\times}$ and
$$\mathcal{L}(m) = \{x_1, \ldots, x_n\}$$
with $x_1 < x_2 < \cdots < x_n$, then

the *delta set* of $m$ is
$$\Delta(m) = \{x_i - x_{i-1} | 2 \leq i \leq n\}$$
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# Some Known Results

If $M$ is a Krull monoid with finite divisor class group $\mathrm{Cl}(M) = G$ where $|G| \geq 3$, then

$$\Delta(M) \subseteq \{1, \ldots, \mathcal{D}(G) - 2\}$$

where $\mathcal{D}(G)$ represents Davenport's constant of $G$.

If $\mathrm{Cl}(M) = \mathbb{Z}_n$ and each divisor class of $\mathrm{Cl}(M)$ contains a prime divisor, then $\Delta(M) = \{1, \ldots, n-2\}$.

On the other hand, the exact delta set of a Krull monoid is known in very few other instances, regardless of the knowledge of the distribution of the prime divisors in $\mathrm{Cl}(M)$.

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# More Background

Of fundamental importance in our study of delta sets, will be the following result of Geroldinger:

**Proposition**

*Let $M$ be a commutative cancellative reduced atomic monoid (i.e., $M^\times = \{0\}$). Then*

$$\min \Delta(M) = \gcd \Delta(M).$$

Hence, if $d = \gcd \Delta(M)$ and $|\Delta(M)| < \infty$, then

$$\Delta(M) \subseteq \{d, 2d, \ldots, kd\}$$

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# GOAL

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Given a numerical monoid $S = \langle n_1, n_2, \dots, n_k \rangle$ describe as best we can the set $\Delta(S)$.

**Is this interesting?**

**Example:** $\Delta(\langle 7, 9, 12 \rangle) = \{1\}$

$\Delta(\langle 7, 10, 12 \rangle) = \{1, 2\}$.

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# What is known.

## Proposition

*Let* $M = \langle n_1, \ldots, n_k \rangle$ *be a primitive numerical monoid with* $n_1, \ldots, n_k$ *a minimal system of generators.*

1. $|\Delta(M)| < \infty$ *and* $\min \Delta(M) = \gcd\{n_i - n_{i-1} \mid 2 \leq i \leq k\}$.
2. *If* $M = \langle n, n+k, n+2k, \ldots, n+bk \rangle$, *then* $\Delta(M) = \{k\}$.
3. *For any* $k$ *and* $v$ *in* $\mathbb{N}$ *there exists a three generated numerical monoid* $M$ *with* $\Delta(M) = \{k, 2k, \ldots, vk\}$.
4. *The sequence* $\{\Delta(x)\}_{x \in M}$ *is eventually periodic.*

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# New Definitions

For any numerical monoid $M$, let $S = \{n_1, n_2, \ldots, n_k\}$ be an arbitrary generating set for $M$.

For $x \in M \setminus M^\times$, set
$\mathcal{F}^S(x) = \{(x_1, \ldots, x_k) \subset \mathbb{N}_0^k \mid x = x_1 n_1 + \ldots + x_k n_k\}$.

For $x \in M \setminus M^\times$, the set

$$\mathcal{L}^S(x) = \{l \in \mathbb{N} \mid \exists\ (x_1, \ldots, x_t) \in \mathcal{F}^S(x) \text{ with } l = x_1 + \cdots + x_l\}$$

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If we have $\mathcal{L}^S(x) = \{l_1, \ldots, l_n\}$, with $l_1 < l_2 < \ldots < l_n$, then

$$\Delta^S(x) = \{l_i - l_{i-1} \mid 2 \leq i \leq n\}$$

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# Results

For a primitive $M = \langle n_1, n_2 \rangle$, we examine closely the behavior of the set $\Delta^S(M)$ for $S = \{n_1, n_2, in_1 + jn_2\}$ for $i, j \geq 0$.

**Theorem**

(1) If $S = \{n_1, n_2, n_1 + n_2\}$, then $\Delta^S(M) = \{1, 2, \ldots, n_2 - n_1\}$.

(2) $\Delta^S(M) = \Delta(M)$ if and only if $i + j - 1 = n_2 - n_1$.

(3) We give exact conditions for when $|\Delta^S(M)| = 1$.

(4) If $i + j = 2$, then $\Delta^S(M) = \{1, 2, ..., k\}$ for some $k$.

(5) If $\Delta^S(M) = \{1, k\}$, then $k = 2$.

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# An Important Tool

**Proposition**

$\mathsf{gcd}(\Delta^S(M)) = \min(\Delta^S(M)) = \mathsf{gcd}\{n_{i+1} - n_i, 1 \leq i < t\}$.

# As Big As You Want

**Theorem**

For any primitive numerical monoid $M$ and all $n \in \mathbb{N}$, there is a finite generating set $S$ such that $|\Delta^S(M)| > n$.

**Sketch of Proof:** We will proceed by showing that for all finite generating sets $S \subset M$, there exists a finite generating set $S' \subset M$ such that $S \subset S'$ and $|\Delta^S(M)| < |\Delta^{S'}(M)|$.

# As Big As You Want

**Theorem**

For any primitive numerical monoid $M$ and all $n \in \mathbb{N}$, there is a finite generating set $S$ such that $|\Delta^S(M)| > n$.

**Sketch of Proof:** We will proceed by showing that for all finite generating sets $S \subset M$, there exists a finite generating set $S' \subset M$ such that $S \subset S'$ and $|\Delta^S(M)| < |\Delta^{S'}(M)|$.

# More of the Proof

Then we can find a series of finite generating sets $S_0, S_1, \ldots, S_n$, where $S_0$ is the minimal generating set, and $|\Delta^{S_i}(M)| \geq i$.
Let $F(M)$ be the Frobenius number of $M$.

Let $d$ be the largest element of $\Delta^{S}(M)$.

Let $s$ be the largest element of $S$.

Let $k$ be the smallest integer such that $\Delta^{S}(M) \subset \bigcup_{i=1}^{k} \Delta^{S}(i)$.

Now choose $m \in \mathbb{N}$ such that $m > \max\{F(M), k, s(d+1)\}$, and let $S' = S \cup \{m\}$. The details of the proof are left to the participant. $\square$

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# As Small As You Want

If we choose our generating set $S$ to include many small elements of $M$ then we can show that $\Delta^S(M)$ is small.

**Theorem**

Let $M$ be a primitive numerical monoid, with minimal generating set $\{n_1, n_2, \ldots, n_k\}$. For all $N \geq 2n_k$, if we let $S = \{m \in M$ such that $m \leq N\}$, then $\Delta^S(M) = \{1\}$.

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# When is the Delta Set a singleton?

We now determine necessary and sufficient conditions for the addition of a non-minimal generator to yield a singleton delta set.

**Theorem**

$|\Delta^S(M)| = 1$ if and only if one of the following two conditions holds:

1. $i + j - 1 = n_2 - n_1$.
2. $j = 0$ and there exists a positive integer $l \leq \lfloor \frac{n_2}{i} \rfloor + 1$ such that $l(i-1) = n_2 - n_1$.

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