Affine convex body semigroups

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Outline

Proportionally modular

Take $a, b \in \mathbb{R}$, $0 < a < b$
Define for every $i \in \mathbb{N}$, $i[a, b] = [ia, ib]$
The set

$$\mathcal{S} = \bigcup_{i=0}^{\infty} i[a, b] = \{0\} \cup [a, b] \cup [2a, 2b] \cup \cdots$$

is a submonoid of $\mathbb{R}$ and
$\mathcal{S} \cap \mathbb{N}$ is a numerical semigroup (it is a proportionally modular numerical semigroup)

We try to repeat this idea to $\mathbb{R}^2$ to obtain fininely generated semigroups

More definitions

$\{a_1, \ldots, a_r\} \subseteq \mathbb{N}^k$,
$S = \langle a_1, \ldots, a_r \rangle = \{\lambda_1 a_1 + \cdots + \lambda_r a_r | \, \lambda_1, \ldots, \lambda_r \in \mathbb{N}\}$.
Every affine semigroup admits a unique minimal generating system
Not every subsemigroup of $\mathbb{N}^k$ is finitely generated

Given $A \subseteq \mathbb{R}^k_+$, define the cone

$$L_{\mathbb{Q}_+}(A) = \left\{ \sum_{i=1}^{p} q_i a_i | p \in \mathbb{N}, q_i \in \mathbb{Q}_+, a_i \in A \right\}.$$

A ray is a line containing the origin O of $\mathbb{R}^k$
To define a ray we only need a point not equal to O

Given $A \subseteq \mathbb{R}^2_+$, denote by τ_1 and τ_2 the extremal rays of $L_{\mathbb{Q}_+}(A)$

Convex body submonoids of $\mathbb{R}^k$

A convex body of $\mathbb{R}^k_+$ is a compact convex subset with nonempty interior

Let F be a convex body of $\mathbb{R}^k_+$, define
$\mathbf{F} = \{X \in \mathbb{R}^k_+ | \text{ there exists } i \in \mathbb{N} \text{ such that } \frac{X}{i} \in F\} \cup \{0\} = \cup_{i=0}^{\infty} F_i$
with $F_i = \{iX | X \in F\}$

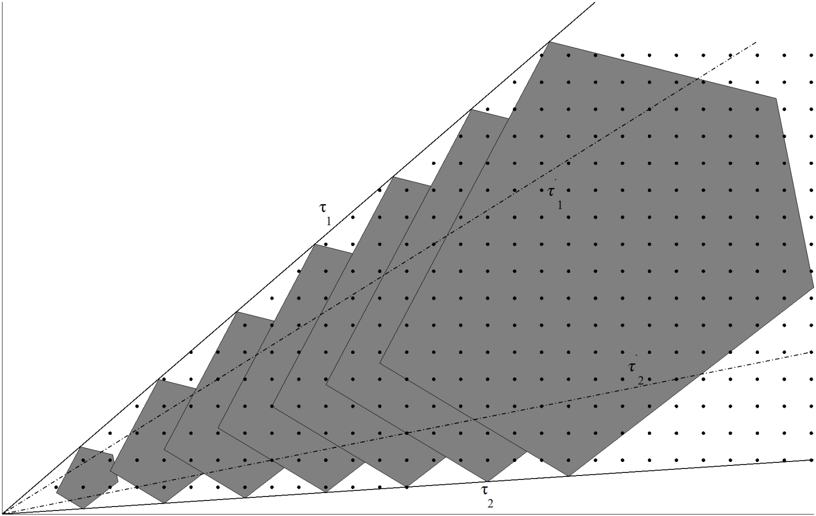
$\mathbf{F}$ is a submonoid of $\mathbb{R}^k$

Convex body semigroups

Define a convex body semigroup as the intersection of a convex body monoid with $\mathbb{N}^k$

In general these semigroups are not finitely generated

We look for conditions to obtain finitely generated subsemigroups of $\mathbb{N}^2$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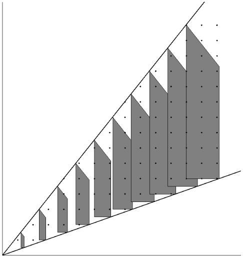
Convex compact polygons

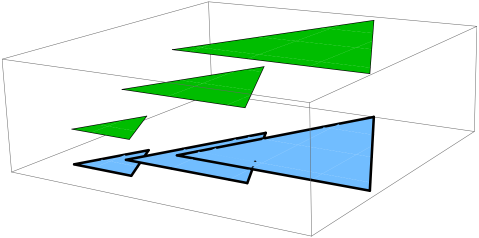
Consider F a convex compact subset of $\mathbb{R}^2_+$ defined by a polygonal closed curve with n vertices
In this case, the compact body semigroup $\mathcal{S}$ defined by F is called a convex polygonal semigroup

Only rational vertices

If all the vertices of F have rational coordinates, then $\mathcal{S}$ is finitely generated

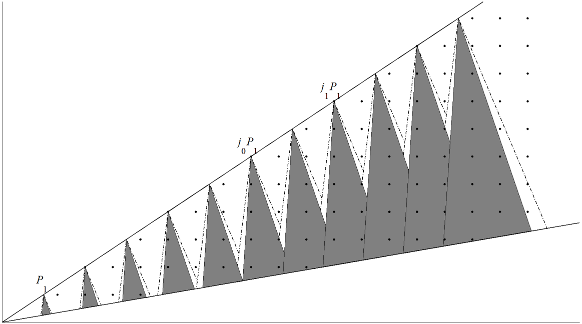


Consider the set $\{0\} \cup \mathrm{Polygon}(\{(v_1, 1), \dots, (v_t, 1)\}) \cup \mathrm{Polygon}(\{(2v_1, 2), \dots, (2v_t, 2)\}) \cup \cdots \subset \mathbb{N}^3$



Triangle

If F is the triangle delimited by $\{P_1, P_2, P_3\}$ with $P_1 \in \mathbb{Q}^2_+$, $P_1 \in \tau_1$, $P_2, P_3 \in \mathbb{R}^2_+ \setminus \mathbb{Q}^2$, $\overline{P_2 P_3} \subset \tau_2$ and τ_2 has rational slope, then $\mathcal{S}$ is finitely generated



Polygon

If τ_1 and τ_2 have rational slopes and $F \cap \tau_1$ and $F \cap \tau_2$ are segments, then $\mathcal{S}$ is finitely generated

Theorem

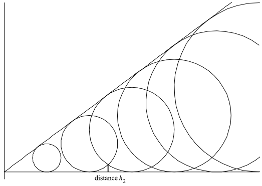
$\mathcal{S}$ is finitely generated if and only if $F \cap \tau_1$ and $F \cap \tau_2$ contain rational points

For all the cases we can compute a system a of generators of $\mathcal{S}$

Definition

Now let F be a convex body delimited by a circle and let $\mathcal{S}$ the convex body semigroup defined by F

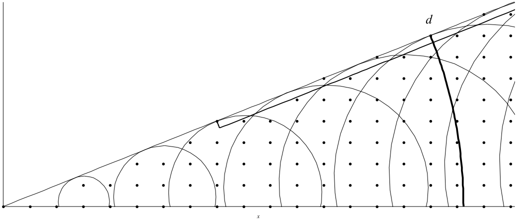
In this case we have the figure



where the limit of the distance h_2 is 0

Theorem

$\mathcal{S}$ is finitely generated if and only if $F \cap \tau_1$ and $F \cap \tau_2$ have rational points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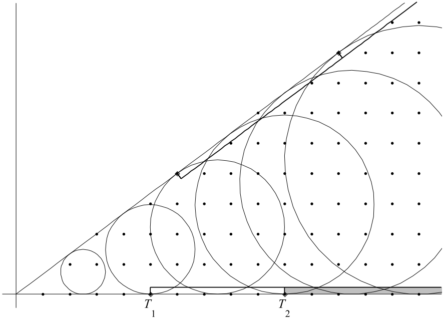
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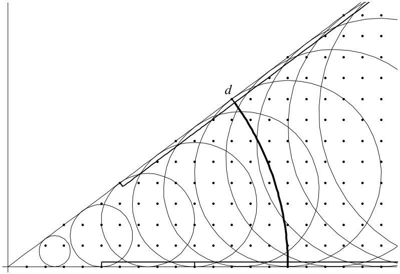
d

circle_semigroup,
http://www.uca.es/dpto/C101/pags-personales/alberto.vigneron/circle_semigroup.rar

J.C. Rosales, P.A. García-Sánchez,
Finitely generated commutative monoids,
Nova Science Publishers, Inc., New York (1999).

J. C. Rosales, P. A. García-Sánchez, J. I. García-García, J. M. Urbano-Blanco,
Proportionally modular Diophantine inequalities,
J. Number Theory 103, 281–294 (2003).

Thanks for your attention!