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On the Betti numbers of some semigroup rings

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based on:

• V. Micale-A. Olteanu, *On the Betti numbers of some semigroup rings*, Le Matematiche vol.67, n.1 (2012).

1. Setup

Let $S = \langle n_1, n_2, \ldots, n_k \rangle$ be a k-generated numerical semigroup and $g(S)$ its Frobenius number.

Let $T(S) = \{x \in \mathbb{Z} \setminus S \mid x + s \in S$ for every $s \in S \setminus \{0\}\}$ be the set of pseudo-Frobenius numbers and $t(S) = |T(S)|$ be the type of S.

S is called symmetric if $t(S) = 1$.

Let $K[S] = K[t^{n_1}, \ldots, t^{n_k}] \subseteq K[t]$ be the semigroup ring associated to S.

$K[S]$ is graded and for each $\alpha \in \mathbb{N}$, $\dim_K K[S]_\alpha$ is 1 or 0, depending of if $\alpha \in S$ or not.

$K[S] = K[X_1, \ldots, X_k]/I = A/I$, $\deg X_i = n_i$, where I is generated by those binomials $X^\beta - X^\gamma$ for which $\sum_i n_i\beta_i = \sum_i n_i\gamma_i$.

The Koszul complex with respect the sequence $X_1, \ldots, X_k$ is a minimal resolution of K.

Tensoring with $K[S]$ and taking the i-th homology gives $\mathrm{Tor}_i^A(K[S], K)$.

$\mathsf{Tor}_i^A(K[S], K)$ is graded with $\mathsf{Tor}_i^A(K[S], K) = \oplus_{s \in S} \mathsf{Tor}_i^A(K[S], K)_s$ and the Betti numbers of $K[S]$ are given by

$$\beta_{i,s}(K[S]) = \mathsf{dim}_K \, \mathsf{Tor}_i^A(K[S], K)_s.$$

2. What is known about $\beta_{i,s}(K[S])$

S is called complete intersection if $K[S]$ is a complete intersection ring.

In this case $\beta_{i,s}(K[S])$ are easily calculated as $\beta_{i,s}(K[S])$ is obtained as a sum of i different degrees of the first syzygy (this is a general fact).

Remark If $k = \text{embdim}(S) \leq 3$, then

$$S \text{ symmetric} \Leftrightarrow S \text{ complete intersection.}$$

• J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3**(1970), 175–193.

Here the author finds the $\beta_{i,s}(K[S])$ for S 3-generated not symmetric.

More precisely, let $S = \langle n_1, n_2, \ldots, n_k \rangle$ and let c_i be the smallest positive integer such that $c_i n_i \in \langle n_1, \ldots, \widehat{n_i}, \ldots, n_k \rangle$ and suppose that $c_i n_i = \sum_{j \neq i} r_{ij} n_i$, then

Theorem (Herzog) Let $S = \langle n_1, n_2, n_3 \rangle$ be not symmetric, then there is a minimal $A = K[X_1, X_2, X_3]$-resolution of $K[S]$:

$$0 \to A[-n_2 c_2 - n_3 r_{13}] \oplus A[-n_3 c_3 - n_2 r_{12}] \to$$

$$\to A[-n_1 c_1] \oplus A[-n_2 c_2] \oplus A[-n_3 c_3] \to A \to K[S] \to 0.$$

• H. Bresinsky, *Symmetric semigroups of integers generated by 4 elements*, Manuscr. Math. **17** (1975), 205–219.

Here the author studies the Betti numbers for the symmetric not complete intersection 4-generated case.

Theorem (Bresinsky) Let $S = \langle n_1, n_2, n_3, n_4 \rangle$ be symmetric but not complete intersection, then there is a minimal $A = K[X_1, X_2, X_3, X_4]$-resolution of $K[S]$:

$$0 \to A[-n_2c_2 - n_3c_3 - n_4r_{14}] \to A^5 \to A^5 \to A \to K[S] \to 0.$$

It is possible to get informations on the Betti numbers of $K[S]$ for other classes of numerical semigroups (such as almost maximal embedding dimension, maximal embedding dimension, 3-generated pseudo symmetric).

• L. G. Fel, *Frobenius problem for semigroups* $S(d_1, d_2, d_3)$, Funct. Analysis and other Math. **1** (2006), 119–157.

• L. G. Fel, *Symmetric numerical Semigroups Generated by Fibonacci and Lucas Triples*, INTEGERS: Electr. J. Comb. number Theory **9** (2009), 106–116.

• J. D. Sally, *Cohen-Macaulay Local Rings of Maximal Embedding Dimensions*, J. Algebra **56** (1976), 168–183.

A good reference on the homology of $K[S]$ is:

- R. Fröberg, *On the homology of semigroup rings*,

$http://cmup.fc.up.pt/cmup/ASA/numsgpsmeeting/slides/froberg.pdf.$

• A. Campillo-C. Marijuan, *Higher order relations for a numerical semigroup*, Journal de Théorie des Nombres de Bordeaux **3** (2) (1991), 249–260.

Here the authors relate the $\beta_{i,s}(K[S])$ to the simplicial complex Δ_s associated to S.

Let $[n] = \{1, \ldots, n\}$ be the vertex set. A simplicial complex Δ on $[n]$ is a collection of subsets of $[n]$ such that if $F \in \Delta$ and $F' \subseteq F$, then $F' \in \Delta$. Each element $F \in \Delta$ is called a face of Δ.

The dimensions of F and Δ are $\dim(F) = |F| - 1$ and $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$.

For each integer i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ and let $K^{|F_i(\Delta)|}$ be a K-vector space whose basis elements e_σ correspond to i-dimensional faces $\sigma \in \Delta$.

The (augmented) chain complex of Δ over K is the complex:

$$0 \longrightarrow K^{|F_{n-1}(\Delta)|} \xrightarrow{\partial_{n-1}} \dots \longrightarrow K^{|F_i(\Delta)|} \xrightarrow{\partial_i} K^{|F_{i-1}(\Delta)|} \longrightarrow \dots$$
$$\xrightarrow{\partial_0} K^{|F_{-1}(\Delta)|} \longrightarrow 0.$$

The boudary maps ∂_i are defined setting $\text{sign}(j, \sigma) = (-1)^{r-1}$ if j is the r^{th} element of the set $\sigma \subseteq [n]$ written in increasing order, and

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \text{sign}(j, \sigma) e_{\sigma \setminus j}.$$

If $i < -1$ or $i > n-1$, then $K^{|F_i(\Delta)|} = 0$ and $\partial_i = 0$ by definition.

Routine check shows that $\partial_i \circ \partial_{i+1} = 0$.

The i^{th}-reduced homology of Δ over K is the K-vector space in homological degree i

$$\tilde{H}_i(\Delta; K) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

If $\Delta = \{\emptyset\}$, then it has homology only in degree -1, where $\tilde{H}_{-1}(\Delta; K) \cong K$.

If $\Delta \neq \{\emptyset\}$, then $\tilde{H}_i(\Delta; K) = 0$ for $i < 0$ or $i > n-1$

If $\Delta \neq \{\emptyset\}$, then $\tilde{H}_0(\Delta; K) \cong K^{r-1}$, where $r = \sharp\{\text{connected components of } \Delta\}$.

If $\Delta \neq \{\emptyset\}$ and $i \geq 1$, then

$$\dim_K(\tilde{H}_i(\Delta; K)) = \sharp\{(i+1)-\text{dimensional holes in } \Delta\}.$$

For $s \in S = \langle n_1, n_2, \ldots, n_k \rangle$, let Δ_s be the simplicial complex with faces $\{n_{i_1}, \ldots, n_{i_j}\}$ such that $s - (n_{i_1} + \cdots + n_{i_j}) \in S$.

Theorem $\beta_{i,s}(K[S]) = \dim_K \tilde{H}_{i-1}(\Delta_s, K)$.

PROOF. $K[S] = K[X_1, \dots, X_k]/I = A/I$ and the Koszul complex on $X_1, \dots, X_k$ is a minimal resolution of K; tensoring with $K[S]$ and taking the i-th homology gives $\operatorname{Tor}_i^A(K[S], K)$. Then it is enough to note that the s-graded part of the Koszul complex tensored with $K[S]$ is isomorphic to the chain complex of Δ_s shifted by one.

See also:

• W. Bruns-J. Herzog, *Semigroups and simplicial complexes*, J. Pure Appl. Algebra **122** (1997), 185–208.

Example Let $S = \langle 3, 4, 5 \rangle$.

$\Delta_0 = \{\emptyset\}$, so $\beta_{0,0}(K[S])$= $\dim_K \tilde{H}_{-1}(\{\emptyset\})$ =1.

Δ_i has not homology for $i = 3, 4, 5, 6, 7$.

Δ_i has $\tilde{H}_0(\Delta_i, K) = K$ for $i = 8, 9, 10$, so $\beta_{1,i}(K[S]) = 1$ for $i = 8, 9, 10$. (These corresponds to the relations $3 + 5 = 4 + 4$, $4 + 5 = 3 + 3 + 3$ and $5 + 5 = 3 + 3 + 4$ in S.)

Δ_i has not homology for $i = 11, 12$.

Δ_i has $\tilde{H}_1(\Delta_i, K) = K$ for $i = 13, 14$, so $\beta_{2,i}(K[S]) = 1$ for $i = 13, 14$.

Δ_i has not homology for $i \geq 15$.

In conclusion, there is still a lot to known about $\beta_{i,s}(K[S])$.

In our paper we are mainly interested on the Betti numbers of $K[T]$ where T is any (among the infinitely many) numerical semigroups for which S is its half. We did it for some classes of numerical semigroups S.

This solves one of the 14 problems proposed by Ralf Fröberg on PRAGMATIC 2011 summer school (Catania, June 20th - July 9th, 2011).

3. The half of a semigroup

Let T be a numerical semigroup and set $\frac{T}{2} = \{x \in \mathbb{N} \mid 2x \in T\}$.

$\frac{T}{2}$ is a semigroup containing T and it is called the half of T.

Theorem (Rosales-García Sánchez) Let S, then there exist infinitely many symmetric numerical semigroups T such that $S = \frac{T}{2}$.

Rosales and García Sánchez show a way to choose the infinitely many T for every $S = \langle n_1, n_2, \ldots, n_k \rangle$.

If $T(S) = \{g_1, \ldots, g_t\}$ and f is an odd integer greater than or equal to $3g(S) + 1$, then $T = \langle 2n_1, 2n_2, \ldots, 2n_k, f - 2g_1, \ldots, f - 2g_t \rangle$ is a symmetric numerical semigroup with Frobenius number f and $S = \frac{T}{2}$.

4. The telescopic case

Due to the fact that telescopic semigroups have a nice structure, they have been intensively studied.

• C. Kirfel, R. Pellikaan, *The minimum distance of codes in an array coming from telescopic semigroups*, IEEE Transactions on information theory, **41**(6), 1995, 1720–1732.

We are interested in computing the Betti numbers of $K[T]$, when T is a symmetric numerical semigroup such that S is its half and S is a telescopic semigroup.

Let $(n_1, \ldots, n_k)$ be a sequence of positive integers with $n_1 < n_2 < \cdots < n_k$ and such that their greatest common divisor is 1.

Define $d_i = \gcd(n_1, \ldots, n_i)$ and $A_i = \{n_1/d_i, \ldots, n_i/d_i\}$ for $i = 1, \ldots, k$.

Let S_i be the semigroup generated by A_i.

If $n_i/d_i \in S_{i-1}$ for $i = 2, \ldots, k$, we call the sequence $(n_1, \ldots, n_k)$ telescopic.

A numerical semigroup is telescopic if it is generated by a telescopic sequence.

Example $S = \langle 6, 10, 11 \rangle$ is telescopic. Indeed

$d_1 = 6$, $d_2 = 2$ and $d_3 = 1$.

$A_1 = \{1\}$, $A_2 = \{3, 5\}$ and $A_3 = \{6, 10, 11\}$.

$S_1 = \mathbb{N}$, $S_2 = \langle 3, 5 \rangle$ and $S_3 = S$.

$n_2/d_2 = 5 \in S_1$ and $n_3/d_3 = 11 = 3 + 3 + 5 \in S_2$.

Remark Telescopic semigroups are symmetric.

Lemma If $S = \langle n_1, \ldots, n_k \rangle$ is telescopic, then $T = \langle 2n_1, \ldots, 2n_k, f - 2g(S) \rangle$ is a telescopic semigroup.

PROOF. We have to show that $(2n_1, \ldots, 2n_k, f-2g(S))$ is a telescopic sequence. The definition is verified by any subsequence of $(2n_1, \ldots, 2n_k)$ since the sequence $(n_1, \ldots, n_k)$ is telescopic by hypothesis. Let $d_{k+1} = \gcd(2n_1, \ldots, 2n_k, f-2g(S))$, then $d_{k+1} = 1$. Since $T_k = S$ and $f - 2g(S) \in T \subseteq \frac{T}{2} = S$, the statement follows.

• K. Watanabe, *Some examples of one dimensional Gorenstein domain*, Nagoya Math. J. **49**(1973), 101–109.

Here the author gives a general procedure to construct new complete intersections from old. Namely, let $S = \langle n_1, \ldots, n_k \rangle$ be complete intersection and let $T = \langle dn_1, \ldots, dn_k, a_1n_1 + \cdots + a_kn_k \rangle$, where $\sum a_i > 1$ and $\gcd(d, a_1n_1 + \cdots + a_kn_k) = 1$. Then T is a complete intersection semigroup.

This says us that any telescopic numerical semigroup is complete intersection.

From the proof of Lemma 1 in Watanabe paper we get the following general result.

Result Let $S = \langle n_1, \ldots, n_k \rangle$ any k-generated numerical semigroup and $a, b \in \mathbb{N}^+$ be such that:

- $a \in S$ with $a \neq n_i$, $i = 1, \ldots, k$
- $\gcd(a, b) = 1$.

Finally, let $T = \langle bn_1, \ldots, bn_k, a \rangle$. If

$$\beta_{1,j}(K[S]) = \begin{cases} 1, & \text{if } j \in \{r_1, \ldots, r_{k-1}\}, \\ 0, & \text{otherwise} \end{cases} ,$$

(that is if $r_1, \ldots, r_{k-1}$ are the degrees of the relations in S), then

$$\beta_{1,j}(K[T]) = \begin{cases} 1, & \text{if } j \in \{br_1, \ldots, br_{k-1}, ba\}, \\ 0, & \text{otherwise.} \end{cases} ,$$

Traslating this in our case where $T = \langle 2n_1, \ldots, 2n_k, f - 2g(S) \rangle$, that is $b = 2$ and $a = f - 2g(S)$, we get

$$\beta_{1,j}(K[T]) = \begin{cases} 1, & \text{if } j \in \{2r_1, \ldots, 2r_{k-1}, 2(f - 2g(S))\}, \\ 0, & \text{otherwise.} \end{cases}$$

Using this last result, we easily compare the first Betti numbers of $K[T_1]$ and $K[T_2]$, where T_1 and T_2 are any two numerical semigroups for which S is their half.

Proposition Let $S = \langle n_1, \ldots, n_k \rangle$ be a telescopic semigroup and $f \geq 3g(S) + 1$ an odd number. Let $T_1 = \langle 2n_1, \ldots, 2n_k, f - 2g(S) \rangle$ and $T_2 = \langle 2n_1, \ldots, 2n_k, f + 2 - 2g(S) \rangle$. Then

$$\beta_{1,j}(K[T_2]) = \begin{cases} \beta_{1,j}(K[T_1]), & \text{if } j \notin \{2(f - 2g(S)), 2(f - 2g(S)) + 4 \\ 0, & \text{if } j = 2(f - 2g(S)), \\ \beta_{1,j}(K[T_1]) + 1, & \text{if } j = 2(f - 2g(S)) + 4. \end{cases}$$

Moreover, by last proposition and by the fact that the graded Betti number β_{ij} in the complete intersection case is obtained as a sum of i different degrees of the first syzygy, we can compare the Betti numbers of $K[T_1]$ and $K[T_2]$, where T_1 and T_2 are any two numerical semigroups for which S is their half.

Indeed, for semplicity, we denote the degrees $2r_i$ of the relations in T_1 (and in T_2), as we saw before, by N_i for $i = 1, \ldots, k-1$ and $[k-1] := \{1, \ldots, k-1\}$.

Corollary Let $S = \langle n_1, \ldots, n_k \rangle$ be a telescopic semigroup and $f \geq 3g(S) + 1$ an odd number. Let $T_1 = \langle 2n_1, \ldots, 2n_k, f - 2g(S) \rangle$ and $T_2 = \langle 2n_1, \ldots, 2n_k, f + 2 - 2g(S) \rangle$. Then

$$\beta_{ij}(K[T_2]) = \begin{cases} \beta_{ij}(K[T_1]), & \text{if } j \in \{N_{t_1} + \cdots + N_{t_i} : \\ & \{t_1, \ldots, t_i\} \subseteq [k-1]\} \\ 0, & \text{if } j \in \{2(f - 2g(S)) + N_{t_1} + \cdots + N_{t_{i-1}} : \\ & \{t_1, \ldots, t_{i-1}\} \subseteq [k-1]\}, \\ \beta_{ij}(K[T_1]) + 1, & \text{if } j \in \{2(f - 2g(S)) + 4 + N_{t_1} + \cdots \\ & \cdots + N_{t_{i-1}} : \{t_1, \ldots, t_{i-1}\} \subseteq [k-1]\}. \end{cases}$$

Remark We can apply the previous result to all the 3-generated symmetric semigroups, since all of them are telescopic.

Indeed any 3-generated symmetric semigroup is of the kind

$$\langle dn_1, dn_2, a_1n_1 + a_2n_2 \rangle$$

where $\gcd(n_1, n_2) = 1$, $a_1 + a_2 > 1$ and $\gcd(d, a_1n_1 + a_2n_2) = 1$.

See for example

- R. Fröberg, *On the homology of semigroup rings*.

5. The 3-generated not symmetric case

Let $S = \langle n_1, n_2, n_3 \rangle$ be not symmetric, $n_1 < n_2 < n_3$.

The type of a 3-generated numerical semigroup is less than or equal to 2 and the type is equal to one if and only if S is symmetric, then

$$T(S) = \{g_1, g_2\}$$

where $g_1 = g(S)$.

Let c_i be the minimal positive integer such that $c_i n_i \in \langle n_j, n_k \rangle$, $i \neq j, \neq k$ and consider $c_i n_i = r_{i,j} n_j + r_{i,k} n_k$.

We can suppose $\gcd(n_i, n_j) = 1$, $i \neq j$, as if $d = \gcd(n_i, n_j)$ Johnson proved that

$$g_1 = g(\langle n_1, n_2, n_3 \rangle) = dg(\langle n_1/d, n_2/d, n_3 \rangle) + (d-1)n_3.$$

The same formula as for g_1 is true for the other pseudo Frobenius number g_2.

Proposition (Rosales-García Sánchez)

$$T(S) = \{(c_3-1)n_3+(r_{1,2}-1)n_2-n_1,\ (c_2-1)n_2+(r_{1,3}-1)n_3-n_1\}.$$

Let $\varphi : K[X, Y, Z] \longrightarrow K[S]$ be the K-algebra homomorphism defined by $\varphi(X) = t^{n_1}$, $\varphi(Y) = t^{n_2}$ and $\varphi(Z) = t^{n_3}$ and let $I_S = \ker \varphi$.

Proposition The ideal I_S is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^{r_{3,1}} & Y^{r_{1,2}} & Z^{r_{2,3}} \\ Z^{r_{1,3}} & X^{r_{2,1}} & Y^{r_{3,2}} \end{pmatrix}.$$

Let $f \geq 3g(S)+1$ be an odd number and $T = \langle 2n_1, 2n_2, 2n_3, f - 2g_1, f - 2g_2 \rangle$.

Furthermore, let us consider $\psi : K[X,Y,Z,U,V] \to K[T]$ the K-algebra homomorphism defined by $\psi(X) = t^{2n_1}$, $\psi(Y) = t^{2n_2}$, $\psi(Z) = t^{2n_3}$, $\psi(U) = t^{f-2g_1}$ and $\psi(V) = t^{f-2g_2}$ and let $I_T = \ker \psi$.

Proposition The ideal I_T is minimally generated by $U^2 - m_1, UV - m_2, V^2 - m_3$ (where m_1, m_2, and m_3 are monomials in $K[X,Y,Z]$) and by the maximal minors of the matrix

$$\begin{pmatrix} X^{r_{3,1}} & Y^{r_{1,2}} & Z^{r_{2,3}} & U \\ Z^{r_{1,3}} & X^{r_{2,1}} & Y^{r_{3,2}} & V \end{pmatrix}$$

if $g_1 = (c_2 - 1)n_2 + (r_{1,3} - 1)n_3 - n_1$ or by the maximal minors of the matrix

$$\begin{pmatrix} X^{r_{3,1}} & Y^{r_{1,2}} & Z^{r_{2,3}} & V \\ Z^{r_{1,3}} & X^{r_{2,1}} & Y^{r_{3,2}} & U \end{pmatrix}$$

if $g_1 = (c_3 - 1)n_3 + (r_{1,2} - 1)n_2 - n_1$.

In the next proposition we get the Betti numbers $\beta_{i,j}(K[T])$ for $i \neq 2$. For simplicity, we denote

$$B_1 = \{2n_2r_{1,2} + 2n_3r_{1,3},\ 2n_1r_{3,1} + 2n_2r_{3,2},\ 2n_1r_{2,1} + 2n_3r_{2,3},\\ 2n_3r_{1,3} + (f - 2g_1), 2n_1r_{2,1} + (f - 2g_1), 2n_2r_{3,2} + (f - 2g_1),\\ 2(f - 2g_1), (f - 2g_1) + (f - 2g_2), 2(f - 2g_2)\},$$

$$B_2 = \{2n_2r_{1,2} + 2n_3r_{1,3},\ 2n_1r_{3,1} + 2n_2r_{3,2},\ 2n_1r_{2,1} + 2n_3r_{2,3},\\ 2n_3r_{2,3} + (f - 2g_1), 2n_1r_{3,1} + (f - 2g_1),\ 2n_2r_{1,2} + (f - 2g_1),\\ 2(f - 2g_1), (f - 2g_1) + (f - 2g_2), 2(f - 2g_2)\},$$

and with $\alpha = 2(n_1 + n_2 + n_3) + (f - 2g_1) + (f - 2g_2) + f$.

Moreover we denote by B_i' the sets $\alpha - B_i := \{\alpha - b \mid b \in B_i\}$ where $i = 1, 2$.

Proposition

$$\beta_{0,j}(K[T]) = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_{4,j}(K[T]) = \begin{cases} 1, & \text{if } j = \alpha, \\ 0, & \text{otherwise} \end{cases}$$

and $\beta_{i,j}(K[T]) = 0$ for every $i \geq 5$ and every j.

Moreover, if $g_1 = (c_2 - 1)n_2 + (r_{1,3} - 1)n_3 - n_1$, then

$$\beta_{1,j}(K[T]) = \begin{cases} 1, & \text{if } j \in B_1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta_{3,j}(K[T]) = \begin{cases} 1, & \text{if } j \in B'_1, \\ 0, & \text{otherwise.} \end{cases}$$

Otherwise, if $g_1 = (c_3 - 1)n_3 + (r_{1,2} - 1)n_2 - n_1$, then

$$\beta_{1,j}(K[T]) = \begin{cases} 1, & \text{if } j \in B_2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_{3,j}(K[T]) = \begin{cases} 1, & \text{if } j \in B_2', \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. For the zero Betti numbers there is nothing to prove.

By $\beta_{4,j} = \dim_K \tilde{H}_3(\Delta_j)$, the only possibility for $\beta_{4,j}$ to be different from zero is for $\beta_{4,j} = 1$ and Δ_j being the empty solid with vertices $\{2n_1, 2n_2, 2n_3, f - 2g_1, f - 2g_2\}$.

This means $j - (2n_1 + 2n_2 + 2n_3 + (f - 2g_1) + (f - 2g_2))$ must not be in T while each time we subtract the sum of 4 different generators of T to j the result must be in T. This can happen only for $j = \alpha$, reminding that f is the Frobenius number of T.

The formula $\beta_{i,j} = \dim_K \tilde{H}_{i-1}(\Delta_j)$ explains why $\beta_{i,j}(K[T]) = 0$ for every $i \geq 5$ and every j.

The results for the first Betti numbers follow by last Proposition.

For the third Betti numbers, we note that, since T is a symmetric numerical semigroup, we have $\beta_{3,j} + \beta_{1,\alpha-j} = \beta_{4,\alpha}$ for every j.

Remark For each concrete example we can also determine $\beta_{2,j}(K[T])$.

Let $S = \langle 3, 5, 7\rangle$ and $T = \langle 6, 10, 14, 15\rangle$. Then $S = \frac{T}{2}$.

The Hilbert series of $K[T]$ is

$$H(K[T], t) = \frac{\sum(-1)^i \beta_{i,j} t^j}{(1-t^6)(1-t^{10})(1-t^{14})(1-t^{15})}.$$

Since we know by last proposition that $\beta_{1,j} = 1$ for $j = 20, 24, 28, 30$ and $\beta_{3,j} = 1$ for $j = 64, 68$, we can compute $\beta_{2,j}$ from the Hilbert series of $K[T]$.

Indeed, we may also write the Hilbert series of $k[T]$ as

$$H(K[T],t) = 1+t^6+t^{10}+t^{12}+t^{14}+t^{15}+t^{16}+t^{18}+t^{20}+t^{21}+t^{22}+\frac{t^{24}}{1-t}.$$

A short calculation gives that $\beta_{2,j} = 1$ for $j = 34, 38, 50, 54, 58$.

Remark Using the last proposition we also compare all but the second Betti numbers of $K[T_1]$ and $K[T_2]$, where T_1 and T_2 are any two numerical semigroups for which S is their half.

6. 4-generated symmetric not complete intersection case

As we saw, Bresinsky shows that a semigroup ring associated to a 4-generated symmetric semigroup S that is not a complete intersection has $\beta_1 = 5$. This helps us to find the total Betti numbers of $K[T]$.

Proposition If $S = \langle n_1, n_2, n_3, n_4 \rangle$ is symmetric semigroup not a complete intersection, and $S = \frac{T}{2}$, then $\beta_1(K[T]) = 6$, $\beta_2(K[T]) = 10$, $\beta_3(K[T]) = 6$,and $\beta_4(K[T]) = 1$. In particular all T have the same total Betti numbers.

PROOF. $T = \langle 2n_1, 2n_2, 2n_3, 2n_4, f - 2g \rangle$ with $g = g(S)$.

Let $\psi : K[X_1, X_2, \ldots, X_5] \to K[T]$ be the K-algebra homomorphism defined by $\psi(X_i) = t^{2n_i}$ for $i = 1, 2, 3, 4$ and $\psi(X_5) = t^{f-2g}$ and let $I_T = \ker \psi$.

By definitions of $g(S)$ and f, we get $X_5^2 - f(X_1, ..., X_4) \in I_T$.

If we compute modulo $X_5^2 - f(X_1, \ldots, X_4)$, then we cannot have a relation $X_5 - g(X_1, \ldots, X_4)$ for degree reason and the other relations $X_5 f(X_1, \ldots, X_4) - X_5 g(X_1, \ldots, X_4)$ follows from the relations $f(X_1, \ldots, X_4) - g(X_1, \ldots, X_4)$ and these are the old relations coming from S.

Since $\beta_1(K[S]) = 5$, then $\beta_1(K[T]) = 6$. Now, since T is symmetric, then $K[T]$ is Gorenstein, so the Betti numbers are symmetric. We have $\beta_4(K[T]) = 1$ and (from the symmetry) $\beta_3(K[T]) = 6$. The alternating sum of the Betti numbers is 0, so $\beta_2(K[T]) = 10$.