On the Apery sets of monomial curves

Tere Cortadellas

University of Barcelona

Iberian meeting on numerical semigroups

Vila Real, July 18^{th} **-** 20^{th}**, 2012**

1 Introduction
- General definitions
- The main goal

2 The strategy
- Tangent cone as a module
- The Apery table
- Reading the structure in the table
- Reading the Hilbert function in the table
- Examples

3 Cohen-Macaulay and Gorenstein tangent cone
- Cohen-Macaulay tangent cone
- Gorenstein tangent cone
- Examples

4 k-Buchsbaum tangent cone
- $k \leq 2$
- Examples

5 Hilbert function
- Non decreasing Hilbert function: The conjecture
- b=4
- Good semigroups

6 References

Let $S = \langle n_1, \ldots, n_b \rangle$ be a numerical semigroup with

- $\{n_1, \ldots, n_b\}$ a minimal system of generators of S
- $n_1 < \cdots < n_b$.

We will call

- $n_1 =: e$ the multiplicity of S and
- b the embedding dimension of S.

Let K be a field and t a variable.

Let

$$K[[S]] := K[[t^s; s \in S]] = K[[t^{n_1}, \dots, t^{n_b}]] \subset k[[t]]$$

be the numerical semigroup ring associated to S. This ring
• is a one dimensional local domain with maximal ideal $\mathfrak{m} = (t^{n_1}, \dots, t^{n_b})$,
• has integral closure the discrete valuation ring $K[[t]]$ (we will denote by v the t-adic valuation in $K[[t]]$),
• is the coordinate ring of the monomial curve in $\mathbb{A}^b_K$ defined by $X_1 = t^{n_1}, \dots, X_b = t^{n_b}$.

Let

$$G(S) := \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

be the tangent cone of K[[S]] (of S).
Our main goal is to relate

arithmetic of S $\longleftrightarrow$ arithmetical properties of $G(S)$.

We recall some facts and fix same notations and definitions

- If I is a fractional ideal of $K[[S]]$ then $v(I)$ is a relative ideal of S.
- For $J \subset I$ fractional ideals of $K[[S]]$, then $\lambda(I/J) = \#v(I) \setminus v(J)$.
- $M = S \setminus \{0\} = v(\mathfrak{m})$ and $nM = M + \overset{n}{\cdots} + M = v(\mathfrak{m}^n)$.

• $s \in S$ has order $\mathrm{ord}(s) = k \Leftrightarrow s \in kM \setminus (k+1)M \Leftrightarrow$
$\Leftrightarrow t^s \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1} \Leftrightarrow 0 \neq (t^s)^* \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$; then

$(t^s)^*$ is the initial form of t^s (of s) in $G(S)$.

If $s = r_1n_1 + \cdots + r_bn_b$ with $\sum_{i=1}^{b} r_i = \mathrm{ord}(s)$, we call this representation a maximal expression of s.

• The element $x = t^e$ generates a minimal reduction of $\mathfrak{m}$ and we can consider the integer

$$r := \min\{r \in \mathbb{N} \mid \mathfrak{m}^{r+1} = x\mathfrak{m}^r\} = \min\{r \mid (r+1)M = e + S + rM\}$$

We will say that r is the reduction number of M.

The following observations are a little sample of the properties that we consider:

- $H(n) := \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \#nM \setminus (n+1)M$
- $(t^s)^* \cdot (t^{s'})^* = 0 \Leftrightarrow \mathrm{ord}(s+s') > \mathrm{ord}(s) + \mathrm{ord}(s')$.
- As $(t^e) \subset \mathfrak{m}$ is a minimal reduction, $(t^e)^*$ is a system of parameters in $G(S)$ and

$$\begin{array}{ll} G(S) & \text{has a non zero divisor} \Leftrightarrow (t^e)^* \text{ is a non zero divisor} \\ & \Updownarrow \\ G(S) & \text{is a Cohen-Macaulay ring} \Leftrightarrow H^0_{G(S)_+}(G(S)) = 0 \end{array}$$

We will put $W = K[[t^e]] \subset A = K[[S]]$ and $F(t^e)$ the fiber cone of $(t^e) \subset A$. This last graded ring

$$F(t^e) := \bigoplus_{n \geq 0} (t^e)^n A/(t^e)^n \mathfrak{m} \cong \bigoplus_{n \geq 0} (t^e)^n W/(t^e)^{n+1} W$$

is a polynomial ring in one variable over K and the extension

$$F(t^e) \hookrightarrow G(S)$$

is finite. Thus, $G(S)$ has a decomposition as a direct sum of a graded finite free $F(t^e)$-module and a torsion module $T(G(S))$ isomorphic to the direct sum of a finite number of modules of the form $(F(t^e)/((t^e)^*)^c F(t^e))(k)$, where k is an integer.

Let $T = \{s \in S;\ \exists c > 0 \text{ with } \mathrm{ord}(s + ce) > \mathrm{ord}(s) + c\}$. Then the map $s \mapsto (t^s)^*$ gives a diagram

$$\begin{array}{ccc} S & \longrightarrow & G(S) \\ \cup & & \cup \\ T & \longrightarrow & T(G(S)) \end{array}$$

If $s \in T$ we say that s is a torsion element and define the torsion order of s as

$$\mathrm{tord}(s) = \min\{c > 0; \mathrm{ord}(s + cn_1) > \mathrm{ord}(s) + c\}.$$

The structure of $G(S)$ just introduced was studied in a more general context by (S. Zarzuela and T.C., 2007); in particular they proved

$$T(G(S)) = \bigcup_{n\geq 0}(0 :_{G(S)} ((t^e)^*)^n = H^0_{G(S)_+}(G(S)).$$

In the next frames we will introduce the Apery table of S (S. Zarzuela and T.C., 2009, S. Zarzuela and T.C., 2011). This table contains the Apery sets with respect to the multiplicity of S of the ideals nM and allows, for instance, to determine an explicit presentation of the tangent cone.

For $n \geq 0$ let (V. Barucci and R. Fröberg, 2006)

$$\mathrm{Ap}(nM) = \{\omega_{n,0} = ne, \ldots, \omega_{n,i}, \ldots, \omega_{n,e-1}\}$$

be the Apery set of nM. Then

$$\mathfrak{m}^n = Wt^{\omega_{n,0}} \oplus \cdots \oplus Wt^{\omega_{n,i}} \oplus \cdots \oplus Wt^{\omega_{n,e-1}}$$

with

- $\omega_{n+1,i} = \omega_{n,i} + \epsilon \cdot e$ where $\epsilon \in \{0, 1\}$,
- $\omega_{n+1,i} = \omega_{n,i} + e$ for $n \geq r$ the reduction number of M.

That is, $t^{\omega_{0,0}}, \ldots, t^{\omega_{0,e-1}}$ is an stacked basis for the free W-modules $\mathfrak{m}^n$.

We define the table

$\mathrm{Ap}(S)$	$\omega_{0,0}$	$\omega_{0,1}$	$\cdots$	$\omega_{0,i}$	$\cdots$	$\omega_{0,e-1}$
$\mathrm{Ap}(M)$	$\omega_{1,0}$	$\omega_{1,1}$	$\cdots$	$\omega_{1,i}$	$\cdots$	$\omega_{1,e-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{Ap}(nM)$	$\omega_{n,0}$	$\omega_{n,1}$	$\cdots$	$\omega_{n,i}$	$\cdots$	$\omega_{n,e-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{Ap}(rM)$	$\omega_{r,0}$	$\omega_{r,1}$	$\cdots$	$\omega_{r,i}$	$\cdots$	$\omega_{r,e-1}$

as the Apery table of S.

We will show how to read the graded structure

$$G(S) \cong \underbrace{\left(F(t^e) \oplus \bigoplus_{i=1}^{e-1} F(t^e)(-d_i)\right)}_{free} \oplus \underbrace{\left(\bigoplus_{i=1}^{e} \bigoplus_{j=1}^{l_i} \frac{F(t^e)}{((t^e)^*)^{c_j^i} F(t^e)}(-b_j^i)\right)}_{torsion}$$

by using the Apery table.

We consider each column in the Apery table as a stair of integers $\omega_0 \leq \cdots \leq \omega_r$ and we say that $\omega_i, \ldots, \omega_{i+k}$ is a landing if $\omega_{i-1} < \omega_i = \cdots = \omega_{i+k} < \omega_{i+k+1}$; i is the beginning and $i+k$ the end of the landing.

We say that the first landing is not a true landing.

Fixed a column in the Apery table:

• the end of the last landing gives an element of a basis of the free submodule of $G(S)$.

• each true landing gives a generator of a cyclic torsion submodule of $G(S)$; the difference of the beginning of the landing and the end of the previous landing reads the corresponding elemental divisor.

The lecture of the Hilbert function in the Apery table it is clear

$$H(n) = \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \\ \#nM \setminus (n+1)M = \\ \#\mathrm{Ap}(nM) \setminus \mathrm{Ap}(n+1)M$$

We provide many explicit examples along the paper. Most of the computations have done by using the NumericalSgps package of GAP (M. Delgado, P. A. García-Sánchez and J. Morais, 2006).

The Apery table for $S = \langle 5, 6, 14 \rangle$ is

$\mathrm{Ap}(S)$	0	6	12	18	14
$\mathrm{Ap}(M)$	5	6	12	18	14
$\mathrm{Ap}(2M)$	10	11	12	18	19
$\mathrm{Ap}(3M)$	15	16	17	18	24
$\mathrm{Ap}(4M)$	20	21	22	23	24

$T = \{14, 19\}$ with $\mathrm{tord}(14) = 2$ and $\mathrm{tord}(19) = 1$.
We will put $F = F(t^5)$ and $x = (t^5)^* \in F$. Then

$$\begin{aligned} G(S) = & \; F \oplus (t^6)^* F \oplus (t^{12})^* F \oplus (t^{18})^* F \oplus (t^{24})^* F \oplus (t^{14})^* F \\ \cong & \; F \oplus F(-1) \oplus F(-2) \oplus F(-3) \oplus F(-4) \oplus F/x^2 F(-1) \end{aligned}$$

and $\mathrm{H}(n) = \{1, 3, 4, 4, 5 \rightarrow\}$.

The Apery table for $S = \langle 7, 8, 17\rangle$ is

$\mathrm{Ap}(S)$	0	8	16	17	25	33	34
$\mathrm{Ap}(M)$	7	8	16	17	25	33	34
$\mathrm{Ap}(2M)$	14	15	16	24	25	33	34
$\mathrm{Ap}(3M)$	21	22	23	24	32	33	41
$\mathrm{Ap}(4M)$	28	29	30	31	32	40	41
$\mathrm{Ap}(5M)$	35	36	37	38	39	40	48
$\mathrm{Ap}(6M)$	42	43	44	45	46	47	48

$T = \{17, 25, 33, 34, 41\}$ and $\mathrm{tord}(s) = 1$ for any $s \in T$.
Writing $F = F(t^7)$ and $x = (t^7)^* \in F$, $G(S)$ has F-module structure

$$F \oplus F(-1) \oplus F(-2) \oplus F(-3) \oplus F(-4) \oplus F(-5) \oplus F(-6) \oplus$$
$$\oplus F/xF(-1) \oplus F/xF(-2) \oplus F/xF(-3) \oplus F/xF(-2) \oplus F/xF(-4)$$

and the Hilbert function is $\mathrm{H}(n) = \{1, 3, 5, 5, 6, 6, 7 \rightarrow\}$.

The following are equivalent

- $G(S)$ is a Cohen-Macaulay ring.
- $G(S)$ is a free $F(t^e)$-module.
- The Apery table of S has no true landings.
- $\mathrm{Ap}(S)$ hasn't torsion elements.

Note that

• the order of the elements in $\mathrm{Ap}(S)$ can be read at the end of the first landing and that,

• in the Cohen-Macaulay case, the reduction number of M coincides with the highest order among these elements.

Write $\mathrm{Ap}(S) = \{\omega_0, \omega_1, \ldots, \omega_{e-1}\}$ with $\omega_0 < \omega_1 < \cdots < \omega_{e-1}$.

Since $G(S)$ is Gorenstein if and only if $G(S)$ is Cohen-Macaulay and S is symmetric and M-pure (L. Bryant, 2010) we have

The following are equivalent

- $G(S)$ is a Gorenstein ring
- The Apery table of S has no true landings and S is symmetric and M-pure.
- It holds
  1. The Apery table of S has no true landings
  2. $\omega_i + \omega_{e-i-1} = \omega_{e-1}$ for all $i = 0, \ldots, e-1$
  3. $\mathrm{ord}(\omega_i) + \mathrm{ord}(\omega_{e-i-1}) = \mathrm{ord}(\omega_{e-1})$ for all $i = 0, \ldots, e-1$

The Apery tables for $\langle 5, 6, 9 \rangle$ and for $\langle 6, 10, 15 \rangle$

S	0	6	9	12	18
M	5	6	9	12	18
$2M$	10	11	14	12	18
$3M$	15	16	19	17	18

S	0	10	15	20	25	35
M	6	10	15	20	25	35
$2M$	12	16	21	20	25	35
$3M$	18	22	27	26	31	35

show that

- the tangent cone of $\langle 5, 6, 9 \rangle$ is Cohen-Macaulay and not Gorenstein,
- the tangent cone of $\langle 6, 10, 15 \rangle$ is Gorenstein.

From now on, this is a joint work with (Raheleh Jafari and Santiago Zarzuela, 2012).

We use the Apery table of the semigroup

- to study systematically the case of embedding dimension 2, and the case of numerical semigroups with a unique Betti element (P. A. García Sánchez, I. Ojeda, J. C. Rosales, 2012).
- to study the k-Buchsbaum property of the tangent cone.
- to study the behavior of the Hilbert function.

$$G(S)\ k\text{-Buchsbaum} \quad \Leftrightarrow G(S)_+^k \cdot H^0_{G(S)_+}(G(S)) = 0$$

$$\Downarrow$$

$$\text{tord}(s) \leq k\ \forall s \in S$$

In this section $S = \langle n_1, n_2, n_3 \rangle$.

We use the Apery table to provide the structure of $G(S)$ when this ring is Buchsbaum or 2-Buchsbaum and, as a consequence we get

- $G(S)$ Buchsbaum $\Leftrightarrow \lambda(T(G(S)) \leq 1$.
- $G(S)$ 2-Buchsbaum $\Leftrightarrow \lambda(T(G(S)) \leq 2$.

(V. A. Sapko, 2001; M. D'Anna, V. Micale and A. Sammartano, 2011; Y. H. Shen, 2011).

The following two lemmas have a key role in the proof of the main result of the present section.

Lemma

Let $x = r_1n_1 + r_2n_2 + r_3n_3$ be a maximal expression of $x \in T$ with $c = \mathrm{tord}(x)$. Let $x + cn_1 = s_1n_1 + s_2n_2 + s_3n_3$ a maximal expression. Then

1. $s_1 = 0$.
2. $r_3 \neq 0$.
3. $s_2 > r_2$.

We order the elements in the Apery set $\mathrm{Ap}(S)$ as follows:

$$\{0, n_2, \ldots, hn_2, \underbrace{n_3, \ldots, n_3 + h_1 n_2}, \ldots, \underbrace{k_S n_3, \ldots, k_S n_3 + h_{k_S} n_2}\}$$

with
• $\mathrm{ord}(kn_3 + j_k n_2) = k + j_k$ for $k = 0, \ldots, k_S$ and $j_k = 0, \ldots, h_k$ $(h_0 = h)$.
• $0 \leq h_{K_S} \leq \cdots \leq h_1 \leq h$.

Lemma

1. $kn_3 \in T \Leftrightarrow kn_3 + h'n_2 \in T$ *and* $tord(kn_3) = tord(kn_3 + h'n_2)$
2. $k < k'$; $kn_3 \in T \Rightarrow k'n_3 \in T$ *and* $tord(k'n_3) \leq tord(kn_3)$

Theorem

For $b = 3$ and the notations introduced

- $G(S)$ *is Cohen-Macaulay* $\Leftrightarrow T = \emptyset \Leftrightarrow k_S n_3 \notin T$.
- $G(S)$ *is Buchsbaum not Cohen-Macaulay* $\Leftrightarrow T = \{k_S n_3\}$.
- $G(S)$ *is 2-Buchsbaum and not Buchsbaum* $\Leftrightarrow$

$$\begin{cases} T = \{k_S n_3, k_S n_3 + n_1\} \text{ or} \\ T = \{k_S n_3, k_S n_3 + n_2\}, \text{ or} \\ T = \{k_S n_3, (k_S - 1) n_3.\} \end{cases}$$

The following three examples are semigroups whose tangents cones are 2-Buchsabum and not Buchsbaum.

For $\langle 5, 6, 14 \rangle$ one has $T = \{0, k_S n_3, k_S n_3 + n_1\} = \{14, 19\}$

For $\langle 8, 11, 18 \rangle$

Ap(S)	0	11	22	33	18	29	36	47
Ap(M)	8	11	22	33	18	29	36	47
Ap($2M$)	16	19	22	33	26	29	36	47
Ap($3M$)	24	27	30	33	34	37	44	47
Ap($4M$)	32	35	42	38	45	41	44	55
Ap($5M$)	40	43	50	46	53	49	52	55

we have $T = \{k_S n_3, k_S n_3 + n_2\} = \{36, 47\}$.

For $\langle 10, 16, 27\rangle$

$\mathrm{Ap}(S)$	0	16	32	48	27	43	59	75	54	81
$\mathrm{Ap}(M)$	10	16	32	48	27	43	59	75	54	81
$\mathrm{Ap}(2M)$	20	26	32	48	37	43	59	75	54	81
$\mathrm{Ap}(3M)$	30	36	42	48	47	53	59	75	64	81
$\mathrm{Ap}(4M)$	40	46	52	58	57	63	69	75	64	91
$\mathrm{Ap}(5M)$	50	56	62	68	67	73	79	85	74	91

we have $T = \{(k_S - 1)n_3, k_S n_3\} = \{54, 81\}$.

It is easy to see that

$$G(S)\; k\text{-Buchsbaum} \Rightarrow \lambda(H^0_{G(S)_+}(G(S))) \leq \frac{k(k+1)^2}{4}$$

- For $k = 2$ we never have the equality.
- Is there a sharp formula for $\lambda(H^0_{G(S)_+}(G(S))$?

The semigroup $\langle 10, 19, 47 \rangle$ has tangent cone 3-Buchsbaum and satisfies the equality in the above formula. That is, $\lambda(H^0_{G(S)_+}(G(S)) = 12$.

The Apery table for $\langle 10, 19, 47 \rangle$ is

$\mathrm{Ap}(S)$	0	19	38	47	66	85	94	113	132	141
$\mathrm{Ap}(M)$	10	19	38	47	66	85	94	113	132	141
$\mathrm{Ap}(2M)$	20	29	38	57	66	85	94	113	132	141
$\mathrm{Ap}(3M)$	30	39	48	57	76	85	104	113	132	141
$\mathrm{Ap}(4M)$	40	49	58	67	76	95	104	123	132	151
$\mathrm{Ap}(5M)$	50	59	68	77	86	95	114	123	142	151
$\mathrm{Ap}(6M)$	60	99	78	87	96	105	114	133	142	161
$\mathrm{Ap}(7M)$	70	79	88	97	106	115	124	133	152	161
$\mathrm{Ap}(8M)$	80	89	98	107	116	25	134	143	152	171
$\mathrm{Ap}(9M)$	90	99	108	117	126	135	144	153	162	171

In this case
$\mathrm{Ap}(S) \cap T = \{n_3, n_3 + n_2, n_3 + 2n_2, 2n_3, 2n_3 + n_2, 2n_3 + 2n_2, 3n_3\}$

Sally's conjecture: the Hilbert function of a one-dimensional Cohen-Macaulay local ring with small enough embedding dimension is non-decreasing.

- The conjecture holds for embedding dimension $b \leq 3$ and also if the tangent cone is Cohen-Macaulay.
- For $b \geq 4$ there are examples of rings with decreasing Hilbert-function.
- For $b = 10$ there are examples of numerical semigroup rings with decreasing Hilbert-function.
- The conjecture is open for numerical semigroup rings with embedding dimension $4 \leq b \leq 9$.

(J. Elias, 1993; J. Elias and J. Martínez-Burruel, 2011; F. Orecchia, 1980; S. K. Gupta and L. G. Roberts, 1983; J. Herzog and R. Waldi, 1975).

Again in the situation of numerical semigroup rings.
Let k be an integer and consider the subsets of $(k-1)M$

$$D_k := \{x;\, \mathrm{ord}(x) = k-1 \text{ and } \mathrm{ord}(x+n_1) > k\} \subset T$$

and

$$C_k := \{y;\, \mathrm{ord}(y) = k \text{ and } y - n_1 \notin (k-1)M\}$$

That is, in the Apery table

$(k-1)M$	$\cdots$	$\cdots$	x	$\cdots$	$\cdots$	y	$\cdots$
kM	$\cdots$	$\cdots$	$x+n_1$	$\cdots$	$\cdots$	y	$\cdots$
$(k+1)M$	$\cdots$	$\cdots$	$x+n_1$	$\cdots$	$\cdots$	$y+n_1$	$\cdots$

Then

$$H(k) - H(k-1) = \#C_k - \#D_k$$

In order to prove that H is not decreasing we will construct an injective map

$$D_k \longrightarrow C_k.$$

We recall that $S = \langle n_1, \ldots, n_b \rangle$. Then, as just we have observed for $b = 3$, if

$$x = \sum_{i=1}^{b} r_i n_i \text{ is a maximal expression of } x \in T$$

with $\mathrm{tord}(x) = c$ and

$$x + cn_1 = \sum_{i=1}^{b} s_i n_i \text{ is a maximal expression of } x + cn_1;$$

then, it holds

- $s_1 = 0$.
- $\sum_{i=3}^{b} r_i \neq 0$.

Observe that the elements in D_k have torsion order 1.
Given an element in D_k the following shows how to obtain an element in C_k.

Lemma (Lemma A)

Assume that $x \in T$ with $\mathrm{tord}(x) = 1$.
Let $x + n_1 = \sum_{i=2}^{b} s_i n_i$ be a maximal expression.
Let $l_x = \mathrm{ord}(x + n_1) - \mathrm{ord}(x) - 1 > 0$.
Then

1. $l_x < \sum_{i=2}^{b-1} s_i$.
2. *Let $y_x = \sum_{i=2}^{b} (s_i - r_i) n_i$, with $0 \le r_i \le s_i$ for all $i = 2, \ldots, b$, and such that $\sum_{i=2}^{b} r_i = l_x$. Then $y_x \in C_k$, where $k = \mathrm{ord}(x) + 1$.*

We use Lemma A to get a very easy proof of Sally's conjecture for numerical semigroup rings with embedding dimension 3

Proposition

The ring $K[[S]]$ associated to a numerical semigroup S with embedding dimension 3 has non-decreasing Hilbert function.

and also for numerical semigroups rings having length of the torsion of the tangent cone equal to one (extending the case Cohen-Macaulay)

Proposition

The ring $K[[S]]$ associated to a numerical semigroup S with $\#T = 1$ has non-decreasing Hilbert function.

Together with Lemma A, the following definition is the principal ingredient in order to prove that the Hilbert function of a numerical semigroup ring of embedding dimension 4 having Buchsbaum tangent cone is not decreasing.

Definition

Assume that $x \in S$. Define $r_x := (r_1, \dots, r_b)$, where $x = \sum_{i=1}^{b} r_i n_i$ is the maximal expression in which

$$r_1 = \max\{r'_1 \mid r'_1 n_1 \text{is part of a maximal expression of } x\},$$
$$r_2 = \max\{r'_2 \mid r_1 n_1 + r'_2 n_2 \text{ is part of a maximal expression of } x\},$$
$$\dots$$
$$r_b = \max\{r'_b \mid \sum_{i=1}^{b-1} r_i n_i + r'_b n_b \text{ is part of a maximal expression of } x\}.$$

If $x = \sum_{i=1}^{b} r_i n_i$ with $r_x = (r_1, \dots, r_b)$ and $x' = \sum_{i=1}^{b} r'_i n_i$ is a sub-representation with $r'_i \le r_i$ then $r_{x'} = (r'_1, \dots, r'_b)$.

If $x, y \in S$ with $r_x = (r_1, \ldots, r_b)$ and $r_y = (s_1, \ldots, s_b)$ we use $r_x \cdot r_y$ to denote the vector $(r_1 \cdot s_1, \ldots, r_b \cdot s_b)$. We denote by 0 the null vector.

Proposition

Assume that $b = 4$ and that for any $x \in T$, $\mathrm{tord}(x) = 1$ and $r_x \cdot r_{x+n_1} = 0$. Then the Hilbert function of $K[[S]]$ is non-decreasing.

Proof.

(An sketch.) Let $x \in D_k$. We consider
$r_x = (0, r_2, r_3, r_4)$,
$r_{x+n_1} = (0, s_2, s_3, s_4)$,
$l_x = \mathrm{ord}(x + n_1) - \mathrm{ord}(x) - 1 < s_2 + s_3$,
$l = \max\{\min\{l_x, s_2 - 1\}, 0\}$.
The map

$$\begin{array}{ccc} D_k & \longrightarrow & C_k \\ x & \mapsto & y_x = (s_2 - l)n_2 + (s_3 - l_x + l)n_3 + s_4 n_4 \end{array}$$

is well defined and injective. □

Taking into account that if $G(S)$ is Buchsbaum then any torsion element is annihilated by $G(S)_+$ we also prove

Proposition

Assume $G(S)$ Buchsbaum. Then

$$x \in T \Rightarrow r_x \cdot r_{x+n_1} = 0.$$

As a consequence of the two above propositions we get

Theorem

The ring $K[[S]]$ associated to a numerical semigroup S with embedding dimension 4 and Buchsbaum tangent cone has non-decreasing Hilbert function.

For $S = < 10, 17, 23, 82 >$ (M. D'Anna, M. Mezzasalma, A. Sammartano) the Apery table is

Ap(S)	0	17	23	34	46	51	68	69	82	85
Ap(M)	10	17	23	34	46	51	68	69	82	85
Ap($2M$)	20	27	33	34	46	51	68	69	92	85
Ap($3M$)	30	37	43	44	56	51	68	69	92	85
Ap($4M$)	40	47	53	54	66	61	68	79	92	85
Ap($5M$)	50	57	63	64	76	71	78	89	102	85
Ap($6M$)	60	67	73	74	86	81	88	99	102	95

It is easy, by using the table, to check that $G(S)$ is Buchsbaum and to read the growth $g(n) = \#C_n - \#D_n$ of the Hilbert function

n	0	1	2	3	4	5	6	...
$g(n)$	0	3	1	2	2	0	1	0...
$H(n)$	1	4	5	7	9	9	10	10...

Definition

We say S is good, if for each element $x \in T$ with $\mathrm{tord}(x) = 1$, there exists $2 \leq i \leq b-1$ such that $x + n_1 = s_i n_i + s_b n_b$ with $r_{x+n_1} = (0, \ldots, s_i, \ldots, s_b)$.

The notion of balanced semigroup has been considered in the case of 4 generated numerical semigroups by (D. P. Patil and G. Tamone, 2011) . Our definition generalizes their definition to any embedding dimension.

Definition

S is called balanced if $n_i + n_j = n_{i-1} + n_{j+1}$ for all $i \neq j \in \{2, \ldots, b-1\}$.

S generated by 3 elements $\Rightarrow$ S good .

Proposition

S balanced $\Rightarrow$ S good .

Using our techniques we can prove

Proposition

The ring $K[[S]]$ associated to a good numerical semigroup S has non-decreasing Hilbert function.

V. Barucci and R. Fröberg , *Associated graded rings of one-dimensional analytically irreducible rings*, J. Algebra **304** (2007), 349–358.

L. Bryant, *Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings*, Comm. Algebra **38** (2010), 2092–2128.

T. Cortadellas and S. Zarzuela, *On the structure of the fiber cone of ideals with analytic spread one*, J. Algebra **317** (2007), 759–785.

T. Cortadellas and S. Zarzuela, *Tangent cones of numerical semigroup rings*, in Viviana Ene, Ezra Miller (Eds.), Combinatorial Aspects of Commutative Algebra, in: Contemp. Math. vol. 502, Amer. Math. Soc., 2009, pp. 45–58.

- T. Cortadellas and S. Zarzuela, *Apery and micro-invariants of a one-dimensional Cohen-Macaulay local ring and invariants of its tangent cone*, J. Algebra **328** (2011), 94–113.
- T. Cortadellas, R. Jafari and S. Zarzuela, *On the Apery sets of monomial curves*, arXiv:1205.4118 (2012).
- M. D'Anna, V. Micale and A. Sammartano, *On the Buchsbaumness of the associated graded ring of a one-dimensional local ring*, Comm. Algebra **37** (2009), 1594–1603.
- M. D'Anna, V. Micale and A. Sammartano, *On the associated graded ring of a semigroup ring*, J. Commut. Algebra, **3** (2011), 147–168.

- M. Delgado, P. A. García-Sánchez and J. Morais, *NumericalSgps - a GAP package, 0.95*, 2006, http://www.gap-system.org/Packages/numericalsgps.
- J. Elías, *The conjecture of Sally on the Hilbert function for curve singularities*, J. Algebra **160** (1993), 42–49.
- J. Elias and J. Martínez-Borruel, *Hilbert polynomials and the intersection of ideals*, Contemp. Math. **555** (2011), 63–70.
- P. A. García-Sánchez, I. Ojeda, and J. C. Rosales, *Affine semigroups having a unique Betti element*, arXiv:1203.4138 (2012).

D. P. Patil and G. Tamone, *CM defect and Hilbert functions of monomial curves*, Journal of Pure and Applied Algebra **215** (2011), 1539–1551.

V. A. Sapko, *Associated graded rings of numerical semigroup rings*, Comm. Algebra, **29** (2001), 4759–4773.

Y. H. Shen, *Tangent cone of numerical semigroup rings with small embedding dimension*, Comm. Algebra, **39** (2011), 1922–1940.