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The Apéry set and the associated graded ring of a numerical semigroup ring

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M. D'Anna - V. Micale - A. Sammartano, When the associated graded ring of a semigroup ring is Complete Intersection, to appear on JPAA

0. Motivations and setup.

$0 < g_1 < g_2 < \cdots < g_\nu \in \mathbb{N}$,

$$S = \langle g_1, \ldots, g_\nu \rangle := \{a_1 g_1 + \cdots + a_\nu g_\nu \mid a_i \in \mathbb{N}, i = 1, \ldots, \nu\}$$

$GCD(g_1, \ldots, g_\nu) = 1 \Longleftrightarrow |\mathbb{N} \setminus S| < \infty$
(in this case $f :=$max$(\mathbb{N} \setminus S)$ is said the Frobenius number)
• $m = g_1$ is said the multiplicity of S

$$R = k[[S]] := k[[t^{g_1}, \ldots, t^{g_\nu}]]$$

R is a one-dimensional domain, local, with maximal ideal $\mathfrak{m} = (t^{g_1}, \ldots, t^{g_n})$ and quotient field $Q = k((t))$.

If $v : k((t)) \longrightarrow \mathbb{Z} \cup \infty$ is the natural valuation, we get

$$v(R) = \{v(r) \mid r \in R \setminus \{0\}\} = S$$

Let G be the associated graded ring of R with respect to $\mathfrak{m}$:

$$G := \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$$

(one-dimensional, graded).

Goal: to study the properties of G (in connection with the properties of S)
For example: Hilbert function; Cohen-Macaulay, Buchsbaum, Gorenstein, Complete Intersection properties...

G is C-M $\iff$ $\overline{t^m}$ is not a zero-divisor (Garcia (1982))

Let $\overline{R} = R/(t^m)$ and

$\overline{G} = \mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{R})$ (0-dimensional, graded)

$$
\begin{array}{ccccc}
G \text{ C.I.} & \Rightarrow & G \text{ Gorenstein} & \Rightarrow & G \text{ C.M.} \\
\Updownarrow & & \Updownarrow & & \\
\begin{array}{c}\overline{G} \text{ C.I.} + \\ G \text{ C.M.}\end{array} & \Rightarrow & \begin{array}{c}\overline{G} \text{ Gorenstein} \\ + \, G \text{ C.M.}\end{array} & & \\
\Downarrow & & \Downarrow & & \\
\overline{G} \text{ C.I.} & \Rightarrow & \overline{G} \text{ Gorenstein} & & \\
\Downarrow & & \Downarrow & & \\
R \text{ C.I.} & \Rightarrow & R \text{ Gorenstein} & \Rightarrow & R \text{ C.M.} \quad \text{(always)}
\end{array}
$$

1. Basic definitions and properties.

- S is symmetric if, $\forall\ x \in \mathbb{Z}$, $x \in S \iff f - x \notin S$.

- S is symmetric $\iff$ R is Gorenstein (Kunz 1971).

We have $k[[x_1, \ldots, x_\nu]]/I \cong R$
(under the homomorphism $\varphi(x_i) = t^{g_i}$)

- R is C.I. $\iff \mu(I) = \nu - 1 \iff$ S is C.I.

- $M = \{s \in S \mid s \neq 0\}$; $m = \min(M) = g_1$ the multiplicity of S;

- $hM = \{s_1 + \cdots + s_h \mid s_i \in M\}$; $\forall h \geq 1$, $(h+1)M \supseteq m + hM$;

- Since $\mathfrak{m}$ is monomial, then $v(\mathfrak{m}^h) = hM$;

- the reduction number $r = r(M)$ of M is the minimal natural number such that $(r+1)M = m + rM$.

- let x such that $v(x) = m (= g_1)$ (e.g. $x = t^m$);
  (x) is a minimal reduction of $\mathfrak{m}$:

$$\exists\ h \in \mathbb{N}_+ \ \text{ such that } \ \mathfrak{m}^{h+1} = x\mathfrak{m}^h$$

the minimum of such integers is the reduction number $r(\mathfrak{m})$;

$$r(\mathfrak{m}) = r(M) = r.$$

2. The Apéry set

We define $\omega_i := min\{s \in S \mid s \equiv i \pmod{m}\}$.

The Apéry set of S with respect to $m = g_1$ is:

$$\text{Ap}_m(S) = \{\omega_0, \ldots, \omega_{m-1}\}$$

We can arrange the elements of $\text{Ap}_m(S)$ in increasing order:

$$\text{Ap}_m(S) = \{w_1 < w_2 < \cdots < w_m\}.$$

We have:

- $w_1 = 0$ and $f + m = w_m$ (where $f =$max$(\mathbb{N} \setminus S)$);
- S symmetric $\Longleftrightarrow w_i + w_{m+1-i} = w_m = f + m \quad \forall\ i = 1, \ldots, m.$

3. Why the Apéry set?

Using the Apéry set one can get information on the following properties for $G = \mathrm{gr}_{\mathfrak{m}}(R)$:

Compl. Int. $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ Buchsbaum and on the Hilbert function of G.

Some recent papers using Apéry set:

Buchsbaum: M. D'Anna - M. Mezzasalma - V. Micale, Y. Shen,
M. D'Anna - V. Micale - A. Sammartano,
T. Cortadellas - S. Zarzuela
T. Cortadellas - R. Jafari - S. Zarzuela

Hilbert function: T. Cortadellas - R. Jafari - S. Zarzuela

4. Cohen-Macaulay property

G C-M $\iff \forall\ s \in hM \setminus (h+1)M \quad s+m \in (h+1)M \setminus (h+2)M$

The blow up of $S = \langle g_1, g_2, \ldots, g_n \rangle$ is the numerical semigroup $S' = \bigcup_{n\geq 1}(nM - nM) = rM - rm = \langle g_1, g_2 - g_1, \ldots, g_n - g_1 \rangle$.
Denote the Apéry set of S' with respect to $m = g_1$ by:

$$\text{Ap}_m(S') = \{\omega'_0, \ldots, \omega'_{m-1}\}$$

Definition. (Barucci-Fröberg) For each $i = 0, 1, \ldots, m-1$ let:
a_i be the only integer such that $\omega'_i + a_i g_1 = \omega_i$;
$b_i = \max\{h \mid \omega_i \in hM\}$.

Theorem. (Barucci-Fröberg)

- $1 \leq b_i \leq a_i \leq r$ and $b_i < a_i \Rightarrow a_i < r$
- $\text{gr}_{\mathfrak{m}}(R)$ is C-M $\iff a_i = b_i$ for each $i = 0, 1, \ldots, m-1$.

5. Gorenstein and Complete Intersection properties

$G = \mathrm{gr}_{\mathfrak{m}}(R)$ $\qquad$ G is C-M $\Leftrightarrow$ $\overline{t^m}$ is not a zero-divisor

$\overline{G} = G/(\overline{t^m})$

- G is Gorenstein (resp. Complete Intersection) $\quad\Longleftrightarrow$
  $\overline{G}$ is Gorenstein (resp. C.I.) and G is Cohen-Macaulay.

Remark. We have $\quad \overline{t^s} = \overline{0}$ in $\overline{G} \Longleftrightarrow t^s \in (t^m) \Longleftrightarrow s - m \in S$.
hence, as k-vector spaces

$$\overline{G} = \langle \overline{t^{\omega_i}} \,|\, \omega_i \in \mathrm{Ap}_m(S) \rangle_k$$

Now, $\overline{G} = \oplus_{h=0}^{n} \overline{G}_h$: can we describe $\overline{G}_h$ in terms of $\mathrm{Ap}_m(S)$?

Definition. Let $s \in S$; if $s \in hM \setminus (h+1)M$ we define $\mathrm{ord}(s) := h$

Remark. We have

$$\overline{G}_h = \langle \overline{t^{\omega_i}} \,|\, \omega_i \in \mathrm{Ap}_m(S) \text{ and } \mathrm{ord}(\omega_i) = h \rangle_k$$

Example.

$S = \langle 8, 10, 15 \rangle = \{0, 8, 10, 15, 16, 18, 20, 23, 24, 25, 26, 28,$
$30, 31, 32, 33, 34, 35, 36, 38, \ldots\},$

$f = 37, \quad m = 8, \quad r = 4,$

$\mathrm{Ap}_8(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$

$$\overline{G} = k \oplus (k\overline{t^{10}} + k\overline{t^{15}}) \oplus (k\overline{t^{20}} + k\overline{t^{25}}) \oplus (k\overline{t^{30}} + k\overline{t^{35}}) \oplus k\overline{t^{45}}$$

6. Gorenstein property (L. Bryant)

Definition. • Let $a, b \in S$, we say that
$a \preceq_M b \iff \exists\ s \in S:\ a + s = b$ and $\mathrm{ord}(a) + \mathrm{ord}(s) = \mathrm{ord}(b)$
• The set of maximal elements of $\mathrm{Ap}_m(S)$ with this partial ordering is denoted with $\mathrm{maxAp_M}(S)$.
• S is M-pure if every $\omega_i \in \mathrm{maxAp_M}(S)$ has the same order.

Proposition. S is M-pure symm. $\iff \mathrm{maxAp_M}(S) = \{f + m\}$
Thm. Let S be M-pure: G is C-M $\iff \max\{\mathrm{ord}(\omega_j)\} = r$

• $\overline{G} = G/(\overline{t^m}) = \oplus_{i=0}^{n} \overline{G}_i$ is Gorenstein $\iff \dim_k \overline{G}_i = \dim_k \overline{G}_{n-i}$

Theorem. • $\overline{G}$ is Gorenstein $\iff S$ is M-pure and symm.;
• G is Gor. $\iff S$ is M-pure symm. and $\mathrm{ord}(f + m) = r$.

Example.

- $S = \langle 8, 10, 15 \rangle \quad \mathrm{Ap}_8(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$

$\mathrm{ord}(30) = 3 > 2 = \mathrm{ord}(15) + \mathrm{ord}(15) \Rightarrow 15 \not\preceq_M 30$,

but $\mathrm{ord}(45) = 4 = 3 + 1 = \mathrm{ord}(30) + \mathrm{ord}(15)$
$\Rightarrow 15 \preceq_M 45$ and $30 \preceq_M 45 \Rightarrow \mathrm{maxAp_M}(S) = \{45\}$.

S is M-pure and symmetric $\Rightarrow \overline{G}$ is Gorenstein. In fact

$$\overline{G} = k \oplus (kv^{10} + kv^{15}) \oplus (kv^{20} + kv^{25}) \oplus (kv^{30} + kv^{35}) \oplus kv^{45}$$

$\mathrm{ord}(f + m) = \mathrm{ord}(37 + 8) = 4 = r \Rightarrow G$ is Gorenstein.

$$
\begin{array}{ccccc}
\begin{array}{c} \textcolor{red}{?} + \\ \mathrm{ord}(f+m) = r \end{array} & \Rightarrow & \begin{array}{c} S \text{ symmetric} \\ + \ M\text{-pure} \\ + \ \mathrm{ord}(f+m) = r \end{array} & \Rightarrow & \begin{array}{c} a_i = b_i \\ \forall i = 1, \ldots, m \end{array} \\
\Downarrow & & \Downarrow & & \\
\textcolor{red}{?} & \Rightarrow & \begin{array}{c} S \text{ symmetric} \\ + \ M\text{-pure} \end{array} & & \\
\Downarrow & & \Downarrow & & \\
S \text{ C.I.} & \Rightarrow & S \text{ symmetric} & &
\end{array}
$$

7. Complete Intersection property (D'A-Mi-Sa)

We have $k[[x_1,\ldots,x_\nu]]/I \cong R$ (under the homom. $\varphi(x_i) = t^{g_i}$)

and $k[x_1,\ldots,x_\nu]/I^* \cong G = \text{gr}_{\mathfrak{m}}(R)$, where $I^* = (in(f) : \ f \in I)$;

here, if $f \in \mathfrak{m}^h \setminus \mathfrak{m}^{h+1}$, then $f = in(f) + g$, with $g \in \mathfrak{m}^{h+1}$.

Since $m = g_1$, $\varphi : x_1 \mapsto t^m$,

$$\overline{G} = G/(\overline{t^m}) \cong k[x_2,\ldots,x_\nu]/J.$$

- $\overline{G}$ is Complete Intersection $\Longleftrightarrow \mu(J) = \nu - 1$.

We would like to determine which shape of $\text{Ap}_m(S)$ implies $\mu(J) = \nu - 1$ and, in this case, the degrees of the generators of J in terms $\text{Ap}_m(S)$.

8. Representations of elements of S

• Let $s \in S$: $s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$; this combination (or the ν-tuple $(\lambda_1, \ldots, \lambda_\nu)$) is a representation of s.

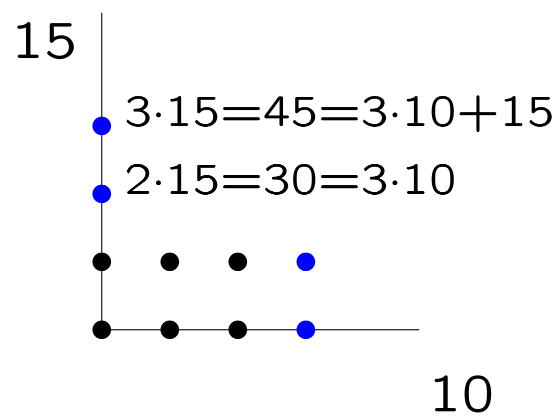
• $\omega_i \in \mathrm{Ap}_m(S)$ can have only representations where g_1 does not appear.

• A representation $s = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_\nu g_\nu$ is maximal if $\lambda_1 + \lambda_2 + \cdots + \lambda_\nu = \mathrm{ord}(s)$.

Definition. For every $i = 2\ldots,\nu$, set:
$\alpha_i := \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}_m(S)\}$;
$\beta_i := \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}_m(S)$ and $\mathrm{ord}(hg_i) = h\}$;
$\gamma_i := \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}_m(S)$, $\mathrm{ord}(hg_i) = h$ and
hg_i has a unique maximal representation$\}$.

Remark. $\forall\ i = 2,\ldots,\nu$, $\gamma_i \leq \beta_i \leq \alpha_i$.

Example. $S = \langle 8, g_2 = 10, g_3 = 15\rangle$;
$\mathrm{Ap}_8(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$.



$\gamma_2 = \beta_2 = \alpha_2 = 3$ and $\gamma_3 = \beta_3 = 1,\ \alpha_3 = 3$.

Definition. For every $i = 2\ldots,\nu$, set:

$\alpha_i := \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}_m(S)\}$;

$\beta_i := \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}_m(S) \;\text{ and }\; \mathrm{ord}(hg_i) = h\}$;

$\gamma_i := \max\{h \in \mathbb{N} \,|\, hg_i \in \mathrm{Ap}_m(S), \;\mathrm{ord}(hg_i) = h \;\text{ and }$

hg_i has a unique maximal representation$\}$.

Example. $S = \langle 8,\; g_2 = 10,\; g_3 = 11,\; g_4 = 12\rangle$;

$\mathrm{Ap}_m(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$.

$2 \cdot 10 \notin \mathrm{Ap}_8(S)$, $\mathrm{ord}(10) = 1 \Rightarrow \alpha_2 = \beta_2 = 1 \Rightarrow \gamma_2 = 1$

$2 \cdot 12 \notin \mathrm{Ap}_8(S)$, $\mathrm{ord}(12) = 1 \Rightarrow \alpha_4 = \beta_4 = 1 \Rightarrow \gamma_4 = 1$

$4 \cdot 11 \notin \mathrm{Ap}_8(S)$, $\mathrm{ord}(33) = 3 \Rightarrow \alpha_3 = \beta_3 = 3$;

$2 \cdot 11 = 10 + 12$ has two maximal representation $\Rightarrow \gamma_3 = 1$.

9. γ-rectangular Apéry set

Definition. $\Gamma = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \gamma_i \right\}$ $\left(|\Gamma| \le \prod_{i=2}^{\nu} (\gamma_i + 1) \right)$.

Theorem. $\mathrm{Ap}_m(S) \subseteq \Gamma$.

Definition. S has γ-rectangular Apéry set if $\mathrm{Ap}_m(S) = \Gamma$.

Theorem. The following conditions are equivalent:
(i) $\mathrm{Ap}_m(S)$ is γ-rectangular;
(ii)

$$f + m = \sum_{i=2}^{\nu} \gamma_i g_i;$$

(iii)

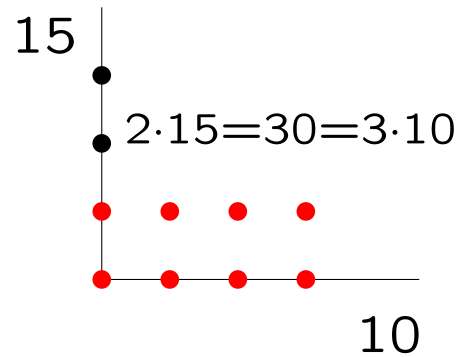
$$m = \prod_{i=2}^{\nu} (\gamma_i + 1).$$

Proposition. $\mathrm{Ap}_m(S)$ γ-rectangular $\Rightarrow$ S is M-pure symmetric.

Examples.

- $S = \langle 8, 10, 15 \rangle$; $\text{Ap}_m(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$,

$\gamma_2 = \beta_2 = 3$, $\gamma_3 = \beta_3 = 1$. $\text{Ap}_8(S)$ is γ-rectangular:



- $S = \langle 8, 10, 11, 12 \rangle$; $\text{Ap}_8(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$,

$\beta_2 = \gamma_2 = 1$, $\beta_3 = 3 > \gamma_3 = 1$, $\beta_4 = \gamma_4 = 1$.

$\text{Ap}_8(S)$ is γ-rectangular:

$\Gamma = \{\lambda_2 \cdot 10 + \lambda_3 \cdot 11 + \lambda_4 \cdot 12 \,|\, 0 \leq \lambda_i \leq 1\}$.

10. The defining ideal J of $\overline{G}$

$$k[x_2,\ldots,x_\nu]/J \cong \overline{G} = G/(\overline{t^m})$$

(under the homomorphism $\varphi(x_i) = \overline{t^{g_i}}$).

$$J = (x_2^{j_2}\cdot\cdots\cdot x_\nu^{j_\nu} - x_2^{h_2}\cdot\cdots\cdot x_\nu^{h_\nu},\ x_2^{\lambda_2}\cdot\cdots\cdot x_\nu^{\lambda_\nu}),$$

where $j_2g_2+\cdots+j_\nu g_\nu = h_2g_2+\cdots+h_\nu g_\nu \in \mathrm{Ap}_m(S)$ are maximal representations and either $\lambda_2g_2+\cdots+\lambda_\nu g_\nu \notin \mathrm{Ap}_m(S)$ or $\lambda_2g_2+\cdots+\lambda_\nu g_\nu \in \mathrm{Ap}_m(S)$ and it is not maximal.

Lemma. • $J \supseteq \tilde{J} = (x_i^{\gamma_i+1} - \rho_i \prod_{j\neq i} x_j^{\lambda_j} :\ i = 2\ldots,\nu)$
where $\beta_i = \gamma_i \ \Rightarrow\ \rho_i = 0$ and $\beta_i > \gamma_i \ \Rightarrow\ \rho_i = 1$
(in this case $(\gamma_i+1)g_i = \sum_{j\neq i}\lambda_j g_j$ are two maximal repr.).

- the smallest pure power of x_i appearing in J is $x_i^{\gamma_i+1}$.

- $\overline{G}$ is CI $\iff$ $J = \tilde{J}$.

Theorem. The following conditions are equivalent:

(i) G is Complete Intersection;
(ii) $\mathrm{Ap}_m(S)$ is γ-rectangular and G is Cohen-Macaulay;
(iii) $\mathrm{Ap}_m(S)$ is γ-rectangular and $r = \mathrm{ord}(f+m)$;

Proof. (ii) $\Rightarrow$ (i) G is C-M, hence G CI $\Longleftrightarrow$ $\overline{G}$ CI.

$\overline{G} = \langle \overline{t^{\omega_i}} \,|\, \omega_i \in \mathrm{Ap}_m(S) \rangle_k$ and $J \supseteq \tilde{J}$, hence

$$(*)\ m = |\mathrm{Ap}_m(S)| = \dim_k(\overline{G}) \leq \dim_k(k[x_2, \ldots, x_\nu]/\tilde{J}) = \prod_{i=2}^{\nu} (\gamma_i + 1)$$

$\mathrm{Ap}_m(S)$ γ-rectangular $\Longleftrightarrow$ $m = \prod_{i=2}^{\nu}(\gamma_i + 1)$;
thus in $(*)$ we have all equalities $\Rightarrow$ $J = \tilde{J}$ $\Longleftrightarrow$ $\overline{G}$ CI.

Examples. • $S = \langle 8, 10, 11, 12 \rangle$; $\text{Ap}_m(S)$ γ-rectangular
$\beta_2 = \gamma_2 = 1$, $\beta_3 > \gamma_3 = 1$, $\beta_4 = \gamma_4 = 1$;

$r = 3 = \text{ord}(33) \Rightarrow G$ C-M. Hence G is CI.
The defining ideals are:
$I = (x_2^2 - x_1x_4, x_3^2 - x_2x_4, x_1^3 - x_4^2)$, $I^* = (x_2^2 - x_1x_4, x_3^2 - x_2x_4, x_4^2)$,

$$J = (x_2^2, x_3^2 - x_2x_4, x_4^2).$$

Note that $x_3^2 - x_2x_4 \quad \longleftrightarrow \quad 2 \cdot 11 = 10 + 12$

• $S = \langle 8, 10, 15 \rangle$; $\text{Ap}_m(S)$ is γ-rectangular;
$\beta_2 = \gamma_2 = 3$, $\beta_3 = \gamma_3 = 1$;

$r = 4 = \text{ord}(45) \Rightarrow G$ C-M. Hence G is CI.
The defining ideals are: $I = (x_2^4 - x_1^5, x_2^3 - x_3^2)$, $I^* = (x_2^4, x_3^2)$,

$$J = (x_2^4, x_3^2).$$

Theorem. Let $S = \langle g_1, g_2, g_3 \rangle$ (with $g_1 < g_2 < g_3$).
Then $\beta_2 = \gamma_2$, $\beta_3 = \gamma_3$ and the following conditions are equivalent:

(i) $\mathrm{Ap}_m(S)$ is γ-rectangular;
(ii) S is M-pure symmetric.

Corollary. Let $R = k[[t^{g_1}, t^{g_2}, t^{g_3}]]$.
G is Gorenstein $\Longleftrightarrow$ G is CI (and the ideal J is monomial).

$$
\begin{array}{ccccc}
\begin{array}{c}\mathsf{Ap}_m(S)\ \gamma\text{-rectangular}\\ +\ \mathsf{ord}(f+m)=r\end{array} & \Rightarrow & \begin{array}{c}S\ \text{symmetric}\\ +\ M\text{-pure}\\ +\ \mathsf{ord}(f+m)=r\end{array} & \Rightarrow & \begin{array}{c}a_i=b_i\\ \forall i=1,\ldots,m\end{array}\\
\Downarrow & & \Downarrow & & \\
\mathsf{Ap}_m(S)\ \gamma\text{-rectangular} & \Rightarrow & \begin{array}{c}S\ \text{symmetric}\\ +\ M\text{-pure}\end{array} & & \\
\Downarrow & & \Downarrow & & \\
S\ \text{C.I.} & \Rightarrow & S\ \text{symmetric} & &
\end{array}
$$

Def. $A = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \alpha_i \right\}$ $B = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \beta_i \right\}$

Remark. $\mathrm{Ap}_m(S) \subseteq \Gamma \subseteq B \subseteq A$.

Definition. S has α- (resp. β-)rectangular Apéry set if $\mathrm{Ap}_m(S) = A$ (resp. $\mathrm{Ap}_m(S) = B$).

Theorem. The following conditions are equivalent:

(i) $\mathrm{Ap}_m(S)$ is α (resp. β)-rectangular;

(ii)

$$f + m = \sum_{i=2}^{\nu} \alpha_i g_i \ (\text{resp.} \beta_i g_i);$$

(iii)

$$m = \prod_{i=2}^{\nu} (\alpha_i + 1) \ (\text{resp.} (\beta_i + 1)).$$

Prop. $\mathrm{Ap}_m(S)$ α-rect. $\Rightarrow$ $\mathrm{Ap}_m(S)$ β-rect. $\Rightarrow$ $\mathrm{Ap}_m(S)$ γ-rect.

$$
\begin{array}{ccccccc}
\begin{array}{c}\mathrm{Ap}_m(S)\\ \alpha\text{-rect.}\end{array} & \Rightarrow & \begin{array}{c}\mathrm{Ap}_m(S)\\ \beta\text{-rect.}\end{array} & \Rightarrow & \begin{array}{c}\mathrm{Ap}_m(S)\\ \gamma\text{-rect.}\end{array} & \Rightarrow & \begin{array}{c}S \text{ symm.}\\ + \ M\text{-pure}\end{array} \\
\Updownarrow & & & & \Downarrow & & \Downarrow \\
\begin{array}{c}S \text{ symm. } +\\ \mathrm{Ap}_m(S) \text{ unique}\\ \text{expression}\end{array} & \Rightarrow & S \text{ free} & \Rightarrow & S \text{ C.I.} & \Rightarrow & S \text{ symm.}
\end{array}
$$

Thanks for the attention!