Filtrations in One-Dimensional Analytically Irreducible Noetherian Local Domains

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Iberian Meeting on Numerical Semigroups - Vila Real 2012
July 19, 2012

Outline

Setting: Rings

Discrete valuation ring

- V is a rank-one DVR with maximal ideal tV and field of fractions K
- $v(f) = \min\{i \mid c_i \neq 0\}$, where $f = \sum c_i t_i \in V$, is the valuation.

Definition

Let $\mathcal{V} = \mathcal{V}(V)$ be the collection of subrings R of V that satisfy the following properties:

- R has field of fractions K and the integral closure of R is V
- V is a finitely generated R-module
- R and V have the same residue field
- $S = \{v(r) \mid r \in R \setminus \{0\}\}$ is a numerical semigroup

Setting: Filtrations

Definitions

- A filtration $F = \{F_i\}_{i=0}^{\infty}$ in a ring R is collection of ideals such that $R = F_0 \supset F_1 \supset F_2 \supset \cdots$ and $F_i F_j \subset F_{i+j}$
- F is good if $JF_n = F_{n+1}$ for $n \gg 0$ whenever J is reduction of F_1
- The multiplicity of F, $e(F)$, is the multiplicity of the ideal F_1
- The blowup ring of F is $B = \bigcup (F_i :_K F_i)$

Definitions

- An ideal $\mathcal{I}$ of a semigroup S is a subset such that $\mathcal{I} + S \subset \mathcal{I}$
- A filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i=0}^{\infty}$ in S is collection of ideals such that $S = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ and $\mathcal{F}_i + \mathcal{F}_j \subset \mathcal{F}_{i+j}$
- $\mathcal{F}$ is good if $e + \mathcal{F}_n = \mathcal{F}_{n+1}$ for $n \gg 0$ where $e = \min\{u \mid u \in \mathcal{F}_1\}$ is the multiplicity
- The blowup semigroup of $\mathcal{F}$ is $\mathcal{B} = \bigcup (\mathcal{F}_i -_{\mathbb{Z}} \mathcal{F}_i)$

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The Induced Semigroup Filtration

The induced semigroup filtration

Let F be a good filtration in $R \in \mathcal{V}$. Then $\mathcal{F} = \{\mathcal{F}_i\}$ such that $\mathcal{F}_i = v(F_i)$ is a good filtration of $S = v(R)$. Moreover,

- $e(F) = e(\mathcal{F})$
- $v(B(F)) = \mathcal{B}(\mathcal{F})$
- The reduction numbers of the two filtrations are equal

A promising start

Much of the basic information is passed to the induced filtration.

Associated Graded Rings

Definition

The associated graded ring of F is $\mathrm{gr}_F(R) = \bigoplus_{i=0}^{\infty} F_i/F_{i+1}$

Proposition (Cohen-Macaulay) [Huckaba and Marley, 1997]

Let $R \in \mathcal{V}$ and F a good filtration with principal red. J, then TFAE

- $\mathrm{gr}_F(R)$ is Cohen-Macaulay
- $JF_i = J \cap F_{i+1}$ for all $i \geq 0$

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Proposition (Gorenstein) [Heinzer et. al., 2009]

Let $R \in \mathcal{V}$ be Gorenstein and $\mathrm{gr}_F(R)$ Cohen-Macaulay, then TFAE

- $\mathrm{gr}_F(R)$ is Gorenstein
- $\ell[(F_i + J) \setminus (F_{i+1} + J)] = \ell[(F_{r-i} + J) \setminus (F_{r-i+1} + J)]$ for all $i \geq 0$

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Additivity and Supersymmetry in Semigroups

Definition

A good filtration $\mathcal{F}$ in a semigroup S is additive if $e + \mathcal{F}_i = (e + S) \cap \mathcal{F}_{i+1}$ for all $i \geq 0$

Definition

An additive filtration $\mathcal{F}$ in a symmetric semigroup S is supersymmetric if

$$\#[\mathcal{F}_i \cup (e+S) \setminus \mathcal{F}_{i+1} \cup (e+S)] = \#[\mathcal{F}_{r-i} \cup (e+S) \setminus \mathcal{F}_{r-i+1} \cup (e+S)]$$

for all $i \geq 0$

Connecting These Ring and Semigroup Properties

Cohen-Macaulay and Additivity

Let $F = \{F_i\}$ be a good filtration in $R \in \mathcal{V}$, and $\mathcal{F} = \{\mathcal{F}_i\}$ where $\mathcal{F}_i = v(F_i)$. Consider

$$\begin{array}{ccc} v(J \cap F_{i+1}) & \subseteq & (e+S) \cap \mathcal{F}_{i+1} \\ \cup\!\vert & & \cup\!\vert \\ v(JF_i) & = & e + \mathcal{F}_i \, . \end{array}$$

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Notes

- Assume $\mathcal{F}$ is an additive filtration in S
- We get $\mathrm{gr}_F(R)$ is Cohen-Macaulay
- We get an additional property

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Definition

A filtration F in $R \in \mathcal{V}$ is called essentially divisible if there exists a principal reduction J of F such that

$$v(J \cap F_i) = v(J) \cap v(F_i) = (e+S) \cap \mathcal{F}_i$$

for all $i \geq 0$.

Connecting These Ring and Semigroup Properties

Cohen-Macaulay and Additivity

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Conclusion

TFAE

- $\mathrm{gr}_F(R)$ is Cohen-Macaulay and F is essentially divisible
- $\mathcal{F}$ is additive

Connecting These Ring and Semigroup Properties

Gorenstein and supersymmetric: A piece of the puzzle

Under certain hypotheses:
TFAE

- $\mathrm{gr}_F(R)$ is Gorenstein
- $\#[v(F_i + J) \setminus v(F_{i+1} + J)] = \#[v(F_{r-i} + J) \setminus v(F_{r-i+1} + J)]$

TFAE

- $\mathcal{F}$ is supersymmetric
- $\#[\mathcal{F}_i \cup (e+S) \setminus \mathcal{F}_{i+1} \cup (e+S)] = \#[\mathcal{F}_{r-i} \cup (e+S) \setminus \mathcal{F}_{r-i+1} \cup (e+S)]$

A connection

If $v(F_i + J) = v(F_i) \cup (J) = \mathcal{F}_i \cup (e + S)$ for all $i \geq 0$, then the second items in the lists are equivalent.

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Connecting These Ring and Semigroup Properties

An amazing equivalence [Barucci and Froberg, 2010]

TFAE

- $v(F_i \cap J) = v(F_i) \cap v(J) = \mathcal{F}_i \cap (e + S)$ (essentially divisible)
- $v(F_i + J) = v(F_i) \cup (J) = \mathcal{F}_i \cup (e + S)$

Conclusion

TFAE

- $\text{gr}_F(R)$ is Gorenstein and F is essentially divisible
- $\mathcal{F}$ is supersymmetric

Essentially Divisible

Monomial Filtrations

We know that a monomial filtration F in a ring $R \in \mathcal{V}$ is essentially divisible. For example, if R is a numerical semigroup ring with maximal ideal m and F is the m-adic filtration.

Question

Is every good filtration F of a ring $R \in \mathcal{V}$ essentially divisible?

Question

If not, does $\mathrm{gr}_F(R)$ being Cohen-Macaulay imply that F is essentially divisible?

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Outline

Additive Ideals

Definitions

- An ideal $\mathcal{I}$ of a semigroup S is called additive if the $\mathcal{I}$-adic filtration $(S \supset \mathcal{I} \supset 2\mathcal{I} \supset \cdots)$ is additive.
- A semigroup S is additive if the maximal ideal $M = S \setminus \{0\}$ is additive.

Common Ideals

Definition

If $S \subset T$ are semigroups and $\mathcal{I}$ is an ideal of S. Then I is called a common ideal of S and T if $\mathcal{I}+T$ is set-theoretically equal to $\mathcal{I}$ as an ideal of S.

Proposition (Common ideal criteria)

Let $S \subset T$ be semigroups and M the maximal ideal of S. Then the following are equivalent

- M is a common ideal of S and T
- $\mathrm{Ap}(T, e) \setminus \mathrm{Ap}(S, e) + e \subset \max\mathrm{Ap}(S, e)$

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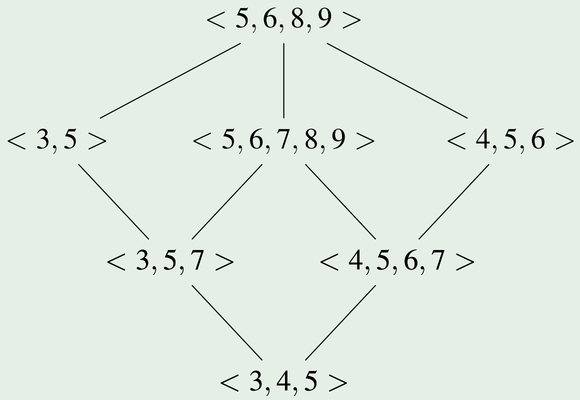
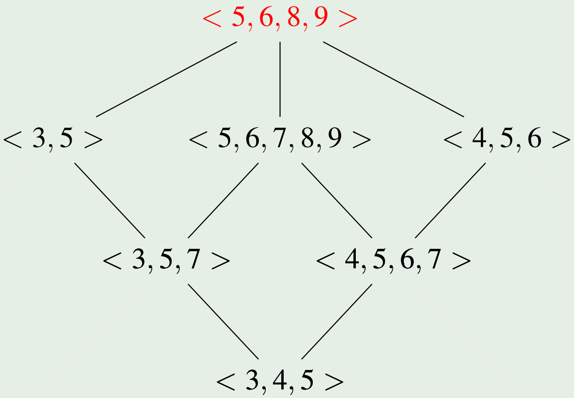
Common Ideal Lattice

Example of a common ideal lattice



Common Ideal Lattice

Example of a common ideal lattice of an additive semigroup



The Common Persistence of Additivity

Theorem (Common Persistence of Additivity) [B.,2009]

Let $S \subset T$ be semigroups, M the maximal ideal of S, and M a common ideal of S and T. Then the following are equivalent

- M is an additive ideal of T
- M is an additive ideal of S and $\mathrm{Ap}(T,e) \setminus \mathrm{Ap}(S,e) + e \subset \mathrm{minAp}(S,e)$

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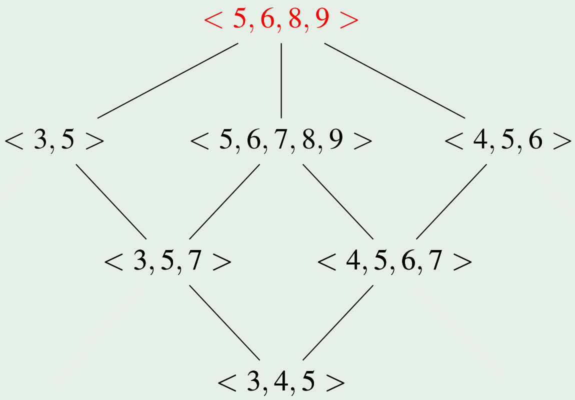
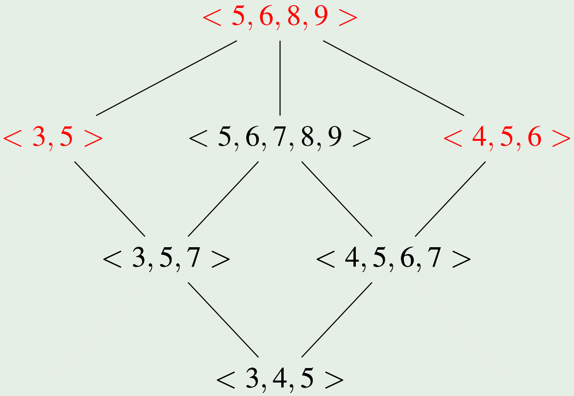


Common Ideal Lattice with Additivity

Example of a common ideal lattice with additivity



Additive Semigroups with multiplicity less than 5

Additive semigroups with $e < 5$ [Barucci, 2006]

1. $S = N_0$
2. $S =< 2, a_2 >$
3. $S =< 3, a_2 >$
4. $S =< 3, a_2, a_3 >$
5. $S =< 4, a_2 >$
6. $S =< 4, a_2, a_3 >$, $a_3 \neq 3a_2 - 4$
7. $S =< 4, a_2, a_3, a_4 >$

Additive Ideals with multiplicity less than 5

Non-principal additive ideals with $e < 5$ [Bryant, 2009]

1. $S = < 2, a_2 >$, $a_2 \neq 3$
   - $(2, a_2) + S$
   - $(4, a_2) + S$
   - $(4, a_2 + 2) + S$
2. $S = < 2, 3 >$
   - $(2, 3) + S$
   - $(3, 4) + S$
   - $(4, 5) + S$
3. $S = < 3, a_2 >$, $a_2 \neq 4$
   - $(3, 2a_2) + S$
   - $(3, a_2) + S$
4. $S = < 3, 4 >$
   - $(3, 2a_2) + S$
   - $(4, 6) + S$
   - $(4, 9) + S$
   - $(3, 4) + S$
5. $S = < 3, a_2, a_3 >$, $a_2 \neq 4$
   - $(3, a_2, a_3) + S$
   - $(3, a_3) + S$
   - $(3, a_2) + S$, if $a_3 < 2a_2 - 3$
6. $S = < 3, 4, 5 >$
   - $(3, 4, 5) + S$
   - $(3, 5) + S$
   - $(4, 5, 6) + S$
   - $(4, 6) + S$
   - $(4, 5) + S$
7. $S = < 4, a_2 >$
   - $(4, 2a_2) + S$
   - $(4, 3a_2) + S$
   - $(4, a_2) + S$
8. $S = < 4, a_2, a_3 >$, a_2, a_3 odd
   - $(4, 2a_2, a_3) + S$
   - $(4, 2a_2) + S$
   - $(4, a_3) + S$, if $a_3 > a_2 + 2$
   - $(4, a_2) + S$, if $a_3 < 3a_2 - 8$
   - $(4, a_3) + S$, if $a_3 = a_2 + 2$
   - $(4, a_2, a_3) + S$
9. $S = < 4, a_2, a_3 >$, a_2 or a_3 even
   - $(4, a_3, a_2 + a_3) + S$
   - $(4, a_2 + a_3) + (S)$
   - $(4, a_2, a_2 + a_3) + S$, if $a_2 \not| a_3 + 4$
   - $(4, a_2, a_3) + S$
10. $S = < 4, a_2, a_3, a_4 >$
    - $(4, a_2, a_3, a_4) + S$
    - $(4, a_3, a_4) + S$
    - $(4, a_4)$
    - $(4, a_2, a_4) + S$, if $a_2 \not| a_3 + 4$
    - $(4, a_2, a_3) + S$, if $a_4 + 4 \notin < a_2, a_3 >$
    - $(4, a_3) + S$, if $a_3 \not| a_4 + 4$
    - $(4, a_2) + S$, if $a_2 \not| a_3 + 4$ and $a_2 \not| a_4 + 4$.

Outline

Factorizations in a Semigroup

Definition

Let $S = \langle a_1, a_2, \ldots, a_d \rangle$ and $u \in S$. If $s = \sum c_i a_i$, then $(c_1, c_2, \ldots, c_t)$ is a factorization of s and $\sum c_i$ is the length of the factorization.

Definitions

- The denumerant of u in S, $d(u, S)$, is the total number of factorizations.
- The maximal denumerant of u in S, $d_{\max}(u, S)$, is the total number of factorization of u with maximal length.

The Maximal Denumerant of a Semigroup

Definition

We call $d_{\max}(S) = \max\{d_{\max}(u; S) \mid u \in S\}$ the maximal denumerant of the S.

Proposition [B., Hamblin, and Jones, to appear in JCA]

The maximal denumerant of a semigroup S is always finite

Comment

The analogous statement for the denumerant does not hold

Setup

- $S = \langle a_1, a_2, \ldots, a_d \rangle$ is a semigroup with $\mathcal{B} = \langle a_1, a_2 - a_1, \ldots, a_d - a_1 \rangle$.
- $\mathcal{B}^{\mathcal{D}}$ is the blowup using $\mathcal{D} = \{a_1, a_2 - a_1, \ldots, a_d - a_1\}$ as the generating set.
- $\max\mathrm{Ap}(S; e)$ denotes the maximal elements of $\mathrm{Ap}(S; e)$ with respect to the partial ordering $u \preceq s$ if $u + u' = s$ for some $u' \in S$.

Theorem [B. and Hamblin]

If S is additive, then $\mathrm{d}_{\max}(S) = \max\{d(f; B^{\mathcal{D}}) \mid f \in \max\mathrm{Ap}(B; e)\}$.

Corollary [B. and Hamblin]

If S is supersymmetric, then $\mathrm{d}_{\max}(S) = d(f(B) + e; B^{\mathcal{D}})$

Maximal Denumerant of an Additive Semigroup

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Computing the Maximal Denumerant

Example

Let $S = \langle 10, 13, 15, 16, 19 \rangle$. Then

- $e = 10$
- $\mathcal{B} = \langle 10, 3, 5, 6, 9 \rangle$
- $f(\mathcal{B}) = 7$
- $\mathrm{d}_{\max}(S) = d(17; \mathcal{B}^{\mathcal{D}}) = 4$
  - $(0, 4, 1, 0, 0)$
  - $(0, 2, 1, 1, 0)$
  - $(0, 1, 1, 0, 1)$
  - $(0, 0, 1, 2, 0)$

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- $\mathrm{d}_{\max}(S) = d(17; \mathcal{B}^{\mathcal{D}}) = 4$
  - $(0, 4, 1, 0, 0)$
  - $(0, 2, 1, 1, 0)$
  - $(0, 1, 1, 0, 1)$
  - $(0, 0, 1, 2, 0)$

Computing the Maximal Denumerant

Example

Let $S = \langle 10, 13, 15, 16, 19 \rangle$. Then

- $e = 10$
- $\mathcal{B} = \langle 10, 3, 5, 6, 9 \rangle$
- $f(\mathcal{B}) = 7$
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Determining Additivity

Definitions

Let $S = \langle a_1, a_2, \ldots, a_d \rangle$ and $\mathcal{B} = \langle a_1, a_2 - a_1, \ldots, a_d - a_1 \rangle$.

- $\text{ord}(w, S)$ is the maximal length of a factorization of $w \in S$.
- $\min \text{ord}(u, \mathcal{B}^{\mathcal{D}})$ is the minimal length of a factorization of $u \in \mathcal{B}$ with respect to the minimal generating set $\mathcal{D} = \{a_1, a_2 - a_1, \ldots, a_d - a_1\}$.

Theorem [Barucci and Froberg, 2006]

TFAE

- S is additive
- $\{w - \text{ord}(w, S)a_1 \mid w \in \text{Ap}(S, a_1)\} \subset \text{Ap}(\mathcal{B}, a_1)$

Theorem [B. and Hamblin]

TFAE

- S is additive
- $\{u + \min \text{ord}(u, \mathcal{B}^{\mathcal{D}})a_1 \mid u \in \text{Ap}(\mathcal{B}, a_1)\} \subset \text{Ap}(S, a_1)$

Improving the Results

Definitions

Let $S = \langle a_1, a_2, \ldots, a_d \rangle$ and $\mathcal{B} = \langle a_1, a_2 - a_1, \ldots, a_d - a_1 \rangle$.

- $\max\mathrm{Ap}_M(S, a_1)$ is set of maximal elements of $\mathrm{Ap}(S, a_1) \setminus \{0\}$ w.r.t. the partial ordering $u \leq w$ if $u + u' = w$ and $\mathrm{ord}(u, S) + \mathrm{ord}(u', S) = \mathrm{ord}(w, S)$.
- $\max\mathrm{Ap}(\mathcal{B}, a_1)$ is set of maximal elements of $\mathrm{Ap}(S, a_1) \setminus \{0\}$ w.r.t. the partial ordering $u \leq w$ if $u + u' = w$.

Theorem [D'anna, Micale and Sammartano, 2010]

TFAE

- S is additive
- $\{w - \mathrm{ord}(w, S)a_1 \mid w \in \max\mathrm{Ap}_M(S, a_1)\} \subset \mathrm{Ap}(\mathcal{B}, a_1)$

Theorem [B. and Hamblin]

TFAE

- S is additive
- $\{u + \min\mathrm{ord}(u, \mathcal{B}^{\mathcal{D}})a_1 \mid u \in \max\mathrm{Ap}(\mathcal{B}, a_1)\} \subset \mathrm{Ap}(S, a_1)$

Thank You