On differential operators of numerical semigroup rings

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Let R be a commutative k-algebra. The ring of differential operators $D(R)$ of R is inductively defined as

$$D^0(R) = \{\Theta_a; a \in R\}$$

where $\Theta_a \colon R \to R$ is the multiplication map $r \mapsto ar$

$$D^n(R) = \{\Theta \in \mathrm{Hom}_k(R,R); [\Theta, D^0(R)] \subseteq D^{n-1}(R)\}$$

where $[\Theta, \Phi] = \Theta\Phi - \Phi\Theta$ is the commutator.

$$D(R) = \cup_{n \geq 0} D^n(R)$$

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The simplest ring of a curve singularity: a $\mathbb{C}$-algebra generated by monomials $\mathbb{C}[[t^{d_1}, \ldots, t^{d_h}]]$, which is a branch of value semigroup $S = \langle d_1, \ldots, d_h \rangle$

The rings of differential operators of such semigroup $\mathbb{C}$-algebras $\mathbb{C}[[S]]$ (or $\mathbb{C}[S]$) have been studied by Perkins, Eriksen, Eriksson. They showed that

- $D(\mathbb{C}[S]) \subseteq D(\mathbb{C}[t, t^{-1}]) = \{f_n\partial^n + \cdots + f_1\partial + f_0; f_i \in \mathbb{C}[t, t^{-1}]\}$
- If $f_n \neq 0$, $f_n\partial^n + \cdots + f_1\partial + f_0 \in D^n(\mathbb{C}[t, t^{-1}])$
- $D(\mathbb{C}[S])$ is a non commutative ring generated as $\mathbb{C}$-algebra by a finite number of differential operators with leading term $f_n\partial^n$, with $f_n \in \mathbb{C}[t, t^{-1}]$.

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Example. $D(\mathbb{C}[t^2, t^3])$ is a $\mathbb{C}$-algebra generated by

$$t^2, t^3, t\partial, t^2\partial, \partial^2 - 2t^{-1}\partial, t\partial^2 - \partial, \partial^3 - 3t^{-1}\partial^2 + 3t^{-2}\partial$$

here for example $\partial t \neq t\partial$. Indeed, if $f \in \mathbb{C}[t^2, t^3]$,

$$\partial t(f) = \partial(tf) = f + tf'$$

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$D(\mathbb{C}[S]) = D(R)$ is a filtered ring, $D^n(R)D^m(R) \subseteq D^{n+m}(R)$ for all m and n, and $D^n(R) \subseteq D^{n+1}(R)$, and its associated graded is $\mathrm{gr}(D(R)) = \oplus_{n \geq 0} D^n(R)/D^{n-1}(R)$.

- $\mathrm{gr}(D(\mathbb{C}[S]))$ is commutative
  In the example $t\partial$ and $\partial t = t\partial + 1$ give the same element of D^1/D^0, so they coincide in $\mathrm{gr}(D(\mathbb{C}[S]))$
- $\mathrm{gr}(D(\mathbb{C}[S])) \subseteq \mathbb{C}[x, y]$. In the example from

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we get

$$\mathrm{gr}(D(\mathbb{C}[t^2, t^3])) = \mathbb{C}[x^2, x^3, xy, x^2y, y^2, xy^2, y^3]$$

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For each $z \in \mathbb{Z}$, define the *valency* of z with respect to a numerical semigroup S as $\mathrm{val}(z) = \#\{s \in S; z + s \notin S\}$.

Example

$$S = \langle 5, 7, 11, 13 \rangle$$

For each $z \in \mathbb{Z}$, define the *valency* of z with respect to a numerical semigroup S as $\text{val}(z) = \#\{s \in S; z + s \notin S\}$.
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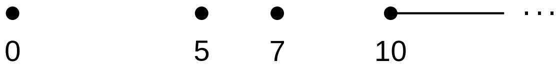

For each $z \in \mathbb{Z}$, define the *valency* of z with respect to a numerical semigroup S as $\mathrm{val}(z) = \#\{s \in S; z + s \notin S\}$. So $\mathrm{val}(z) \geq 0$, for each $z \in \mathbb{Z}$.

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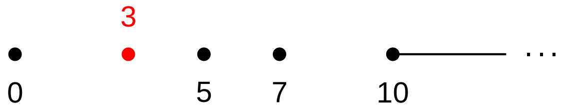

$\mathrm{val}(3) = 2$ because $3 + 0 \notin S$ and $3 + 5 \notin S$

$\mathrm{val}(-3) = 5$, because $-3 + 0 \notin S$, $-3 + 5 \notin S$, $-3 + 7 \notin S$, $-3 + 11 \notin S$, $-3 + 12 \notin S$.

$\mathrm{val}(-3) = \mathrm{val}(3) + 3 = 2 + 3 = 5$

Lemma (P.T. Perkins)

For each $z \in \mathbb{Z}$,

$$val(-z) = val(z) + z$$

Theorem (E. Eriksen, A. Eriksson, independently)

Let $S = \langle d_1, \ldots, d_\nu \rangle$ *be a numerical semigroup,*
let $\pm H(S) = \{\pm h; h \in \mathbb{N} \setminus S\}$
and let $A = \mathrm{gr}(D(\mathbb{C}[S]))$
then A is minimally generated by

$$\{x^{d_1}, \ldots, x^{d_\nu}, y^{d_1}, \ldots, y^{d_\nu}\} \cup \{xy\} \cup \{x^{val(-h)} y^{val(h)}\}_{h \in \pm H(S)}$$

Thus $A = \mathbb{C}[\Sigma]$, where

$$\Sigma = \langle (d_1, 0), \ldots, (d_\nu, 0), (1, 1), (0, d_1), \ldots, (0, d_\nu), \{(\mathrm{val}(-h), \mathrm{val}(h))\} \rangle$$

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Example If $S = \langle 3, 5 \rangle$, then
$H(S) = \{7, 4, 2, 1\}$ and

$\mathrm{val}(7) = 1, \mathrm{val}(4) = 2, \mathrm{val}(2) = 2, \mathrm{val}(1) = 3.$

Thus, if $\mathrm{gr}(D(\mathbb{C}[S])) = \mathbb{C}[\Sigma]$, then Σ is minimally generated by
$(3, 0), (5, 0), (1, 1), (0, 3), (0, 5)$, and....

$(\mathrm{val}(-7), \mathrm{val}(7)) = (8, 1)$, in fact $((\mathrm{val}(-h) = \mathrm{val}(h) + h)$
$(\mathrm{val}(-4), \mathrm{val}(4)) = (6, 2)$
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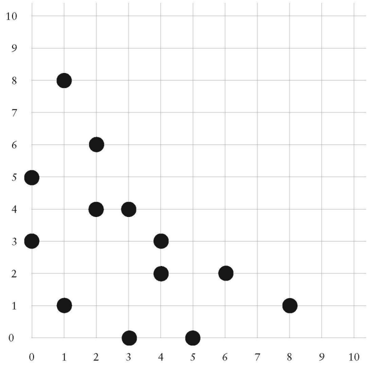
$$(h = \pm 7, \pm 4, \pm 2, \pm 1)$$



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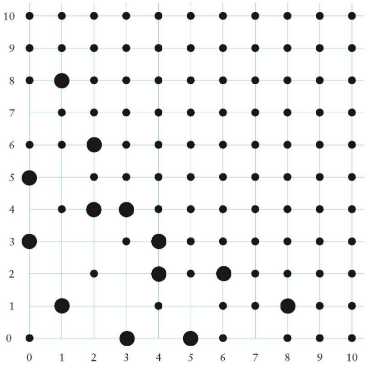

For $z \in \mathbb{Z}$, let

$$\Delta_z = \{(a, b) \in \mathbb{N}^2; a - b = z\}$$

If $s \in \pm S$, since $(1, 1) \in \Sigma$, then $\Delta_s \subset \Sigma$.

For (a, b) in such diagonal Δ_s,

$$\mathrm{val}(a - b) = \mathrm{val}(s) = 0 \text{ or } s$$

In both cases $\mathrm{val}(a - b) \leq b$.

If $h \in \pm H(S)$ and $(a, b) \in \Delta_h$, then

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Proposition (B. - Fröberg)

Let $(a, b) \in \mathbb{N}^2$. Then

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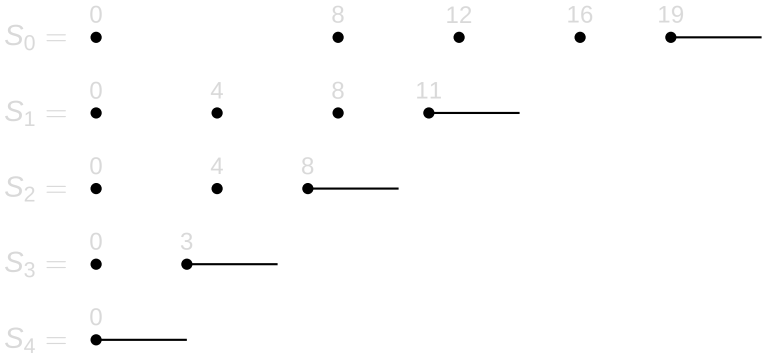
If $s \in S$, set $I(s) = \{n \in S; n \geq s\}$, which is an ideal of S.

A numerical semigroup S is *Arf* if

$$I(s) - s$$

is a semigroup for each $s \in S$. An Arf numerical semigroup:

$$S = \langle 8, 12, 19, 22 \rangle$$



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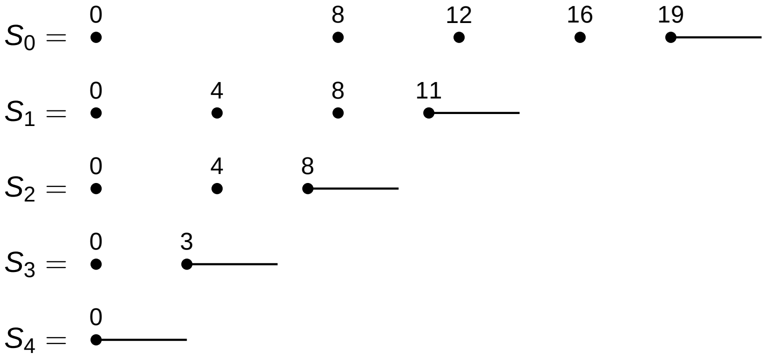


Let S be a numerical semigroup with consecutive blowups

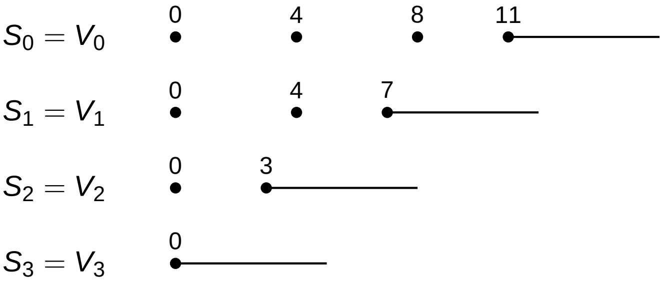
$$S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$$

Lemma (B. - Fröberg)

If S is an Arf semigroup, then $S_i = V_i$, for each i, where $V_i = \{n \in \mathbb{N}; \mathrm{val}(n) \leq i\}$

Example

$$S = \langle 4, 11, 13, 14 \rangle$$



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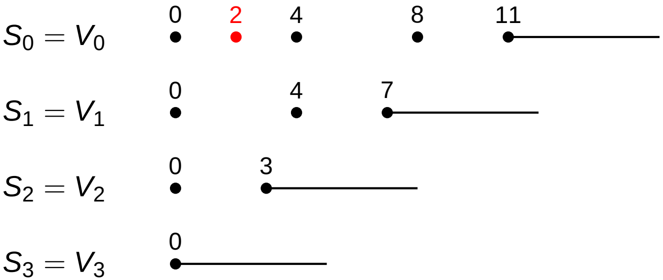
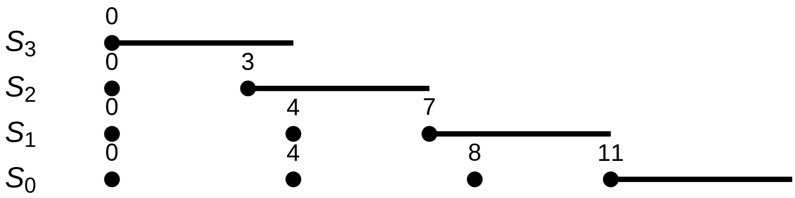


val(2) $= 3$, in fact $2 + 0 \notin S$, $2 + 4 \notin S$ and $2 + 8 \notin S$ and $2 \in S_3 \setminus S_2 = V_3 \setminus V_2$

So, if S is an Arf numerical semigroup, we have an easy recipe to construct $A = \mathrm{gr}(D(\mathbb{C}[S]))$. We know in fact that (if $h \geq k$):

$$x^h y^k \in A \iff h - k \in V_k \iff h - k \in S_k$$

Picturing any monomial $x^h y^k$ with the point (h, k) in the plane, let's see what is $A = \mathrm{gr}(D(\mathbb{C}[S]))$, taking the previous Arf semigroup $S = \langle 4, 11, 13, 14 \rangle$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S_3
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3
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7
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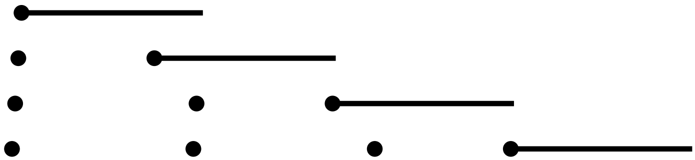
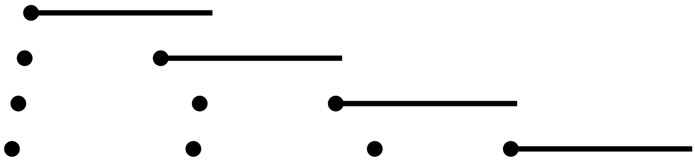
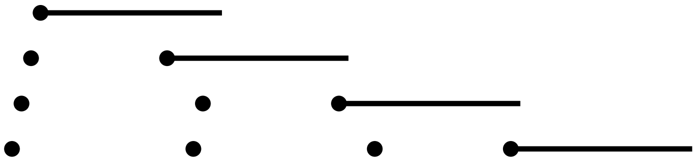
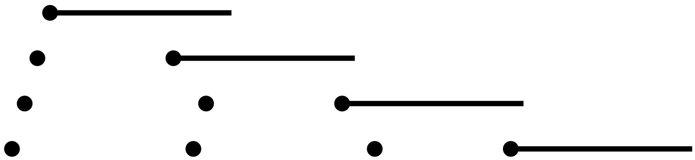
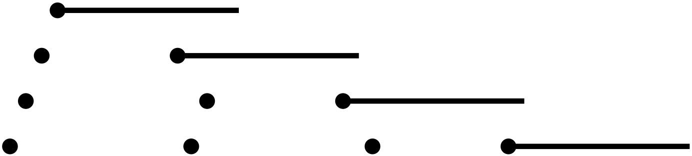
S_0

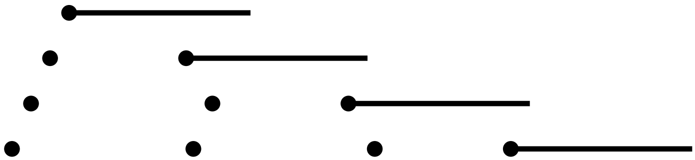


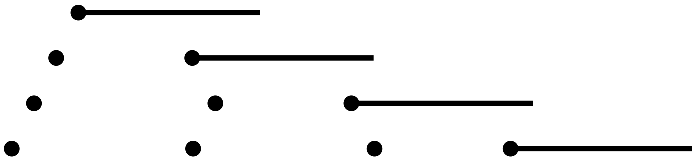


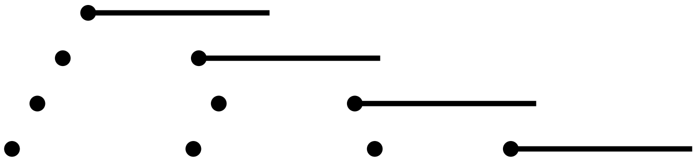


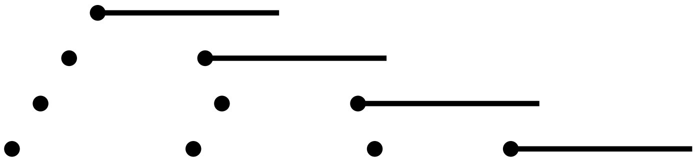


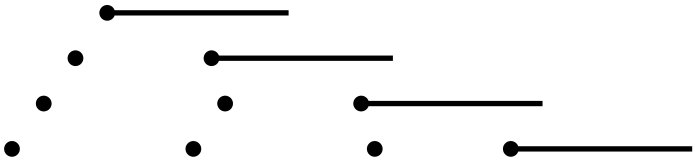


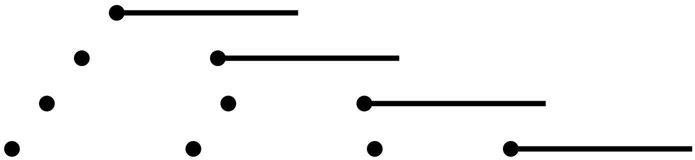


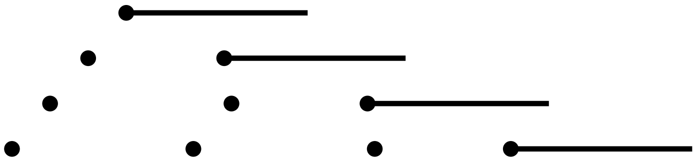


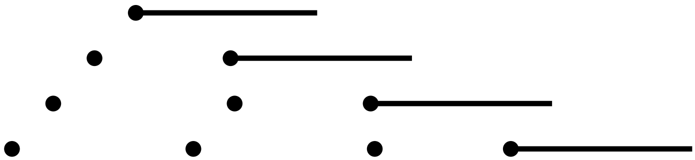


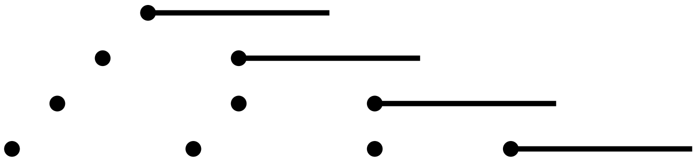


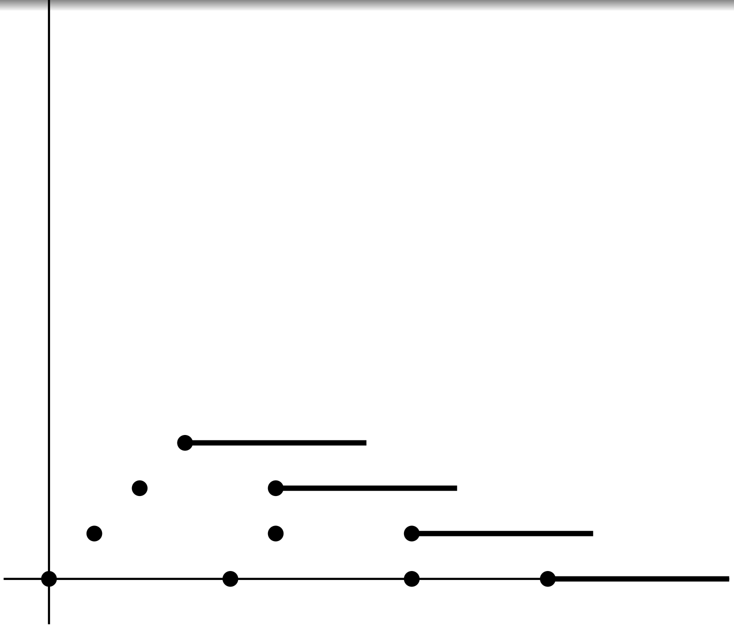


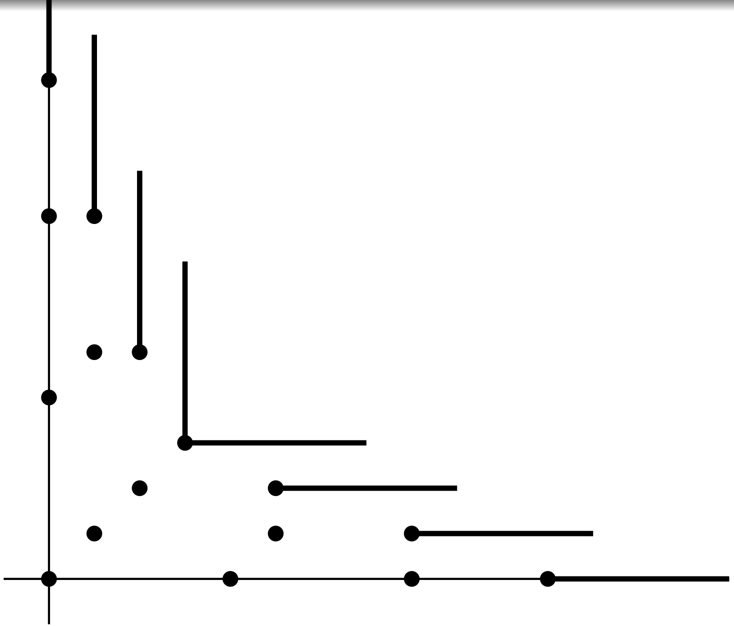












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- an *affine semigroup* (i.e. a subsemigroup of $\mathbb{Z}^d$, for some d)
- *pointed* or *positive* (i.e. the only $\sigma \in \Sigma$ such that $-\sigma \in \Sigma$ is $(0,0)$)
- the quotient group of Σ, gp(Σ) is $\mathbb{Z}^2$
- the normalization of Σ is

$$\overline{\Sigma} = \{x \in \mathrm{gp}(\Sigma);\ mx \in \Sigma,\ \text{for some } m \in \mathbb{N}, m > 1\} = \mathbb{N}^2$$

- the holes of Σ are a finite number, $H(\Sigma) = \overline{\Sigma} \setminus \Sigma$
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$$T(\Sigma) = \{\tau \in \mathrm{gp}(\Sigma); \tau \notin \Sigma, \tau + \Sigma_+ \subseteq \Sigma_+\}$$

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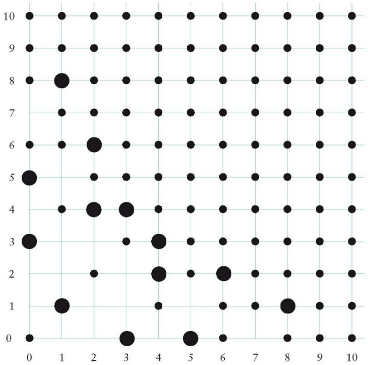

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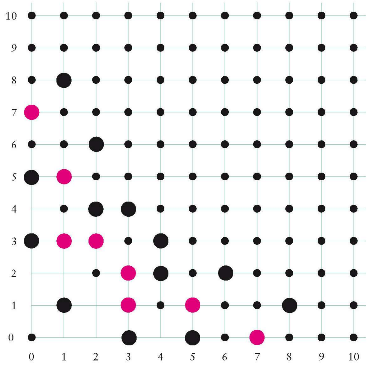

Thus $\mathbb{C}[\Sigma]$ is a two-dimensional Noetherian (non Cohen Macaulay) ring.

Proposition (E. Emtander)

For each affine semigroup Σ, $T(\Sigma)$, the set of pseudoFrobenius numbers, is finite.

Proof. Let $\sigma \in \Sigma \setminus \mathbf{0}$. The semigroup ideal generated by $\sigma + u;\ u \in T(\Sigma)$ is f. g.. If $u_1, u_2 \in T(\Sigma)$, $u_1 \neq u_2$, then $(\sigma + u_1)$ and $(\sigma + u_2)$ are both necessary to generate the ideal, because $(\sigma + u_1) - (\sigma + u_2) = u_1 - u_2 \notin \Sigma$. Thus $T(\Sigma)$ is finite.

If $\sigma \in \Sigma$, the Apery set of Σ with respect to σ is

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$$\sigma_1 \preceq \sigma_2 \ \Leftrightarrow \sigma_1 + \sigma_3 = \sigma_2, \ \text{for some} \ \ \sigma_3 \in \Sigma \qquad (\star)$$

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Let $\Sigma \subset \mathbb{Z}^d$ be a positive affine semigroup and let $0 \neq \sigma \in \Sigma$. Then the following are equivalent for $x \in \mathbb{Z}^d$:
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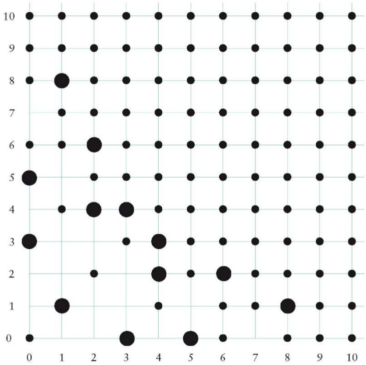

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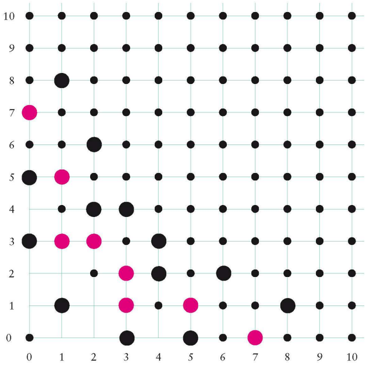

It is well known that, for a numerical semigroup S, the cardinality of $T(S)$ is the CM type of $\mathbb{C}[S]$, i.e. $t = |T(S)|$ is the number of components of a decomposition of a principal ideal as irredundant intersection of irreducible ideals.

Has $|T(\Sigma)|$ a similar meaning in the ring $\mathbb{C}[\Sigma]$?

Let I be a proper ideal of Σ i.e. a proper subset I of Σ such that $I + \Sigma \subseteq I$. I is *irreducible* if it is not the intersection of two ideals which properly contain I. I is *completely irreducible* if it is not the intersection of any set of ideals which properly contain I.

For $x \in \Sigma$, set

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Facts:

The irreducible ideals of Σ are of the following forms:

- $N_{(a,0)} := \Sigma \cap \{(x, y) \in \mathbb{N}^2; x \geq a\}$
- $N_{(0,b)} := \Sigma \cap \{(x, y) \in \mathbb{N}^2; y \geq b\}$
- $I = \Sigma \setminus B(x)$, for some $x \in \Sigma$, which is *completely irreducible* i.e. not the intersection of any set of ideals which properly contain I.

Proposition (B. - Fröberg)

Let I be an ideal of Σ generated by $(a_1, b_1), \ldots, (a_h, b_h)$ and let $a = \min\{a_i\}$, $b = \min\{b_i\}$. Then

$$I = \bigcap_{x \in \max(\Sigma \setminus I)} (\Sigma \setminus B(x)) \cap N_{(a,0)} \cap N_{(0,b)}$$

is the unique irredundant decomposition of the ideal I as intersection of irreducible ideals.

If $I = \sigma + \Sigma$ is principal, then

$$\max(\Sigma \setminus I) = \max(\Sigma \setminus (\sigma + \Sigma)) = \max \mathrm{Ap}_\sigma(\Sigma)$$

So:

Corollary (B. - Fröberg)

If $(0,0) \neq \sigma = (a,b) \in \Sigma$*, then*

$$\sigma + \Sigma = \bigcap_{x \in \max \mathrm{Ap}_\sigma(\Sigma)} (\Sigma \setminus B(x)) \cap N_{(a,0)} \cap N_{(0,b)}$$

is the unique irredundant decomposition of the principal ideal $\sigma + \Sigma$ *as intersection of irreducible ideals.*

Thus the number of irreducible components for a principal ideal is

$$|\max \mathrm{Ap}_\sigma(\Sigma)| + 2 = |T(\Sigma)| + 2 = 2|H(S)| + 2$$

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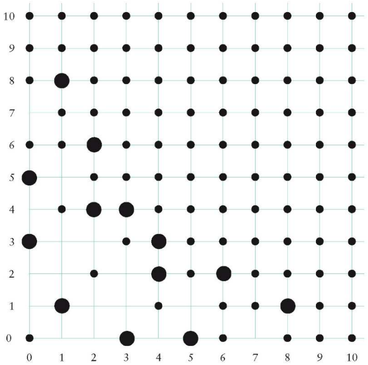

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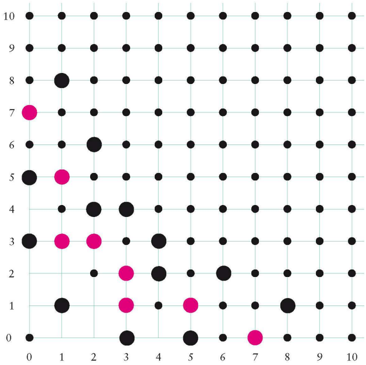

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Corollary

Each principal monomial ideal of $\mathbb{C}[\Sigma]$ is an irredundant intersection of $|T(\Sigma)| + 2 = 2|H(S)| + 2$ irreducible ideals.

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