Feng Rao numbers
(A short history and last conquers)

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Definition of distance. Hamming distance

- Let $x, y \in \mathbb{F}_q^n$ we define the Hamming distance as $d(x, y) = \sharp\{i \colon x_i \neq y_i\}$.
- For a code $C \subseteq \mathbb{F}_q^n$ the Hamming distance of C is defined as the minimal distance for two elements.

$$d = min\{d(c, c') \colon c, c' \in C\}.$$

AG Codes-Goppa bound

- For AG (algebraic-geometric) codes for one point, (valuations of rational functions over a curve with a unique pole P) the most classical bound is de Goppa bound:

$$d \geq m + 2 - 2g$$

- Where m is the order of the pole of a certain rational function, and g is the genus of the curve.

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Weierstrass semigroup

For this type of codes, we can associate a semigroup called Weierstrass semigroup. Now m is an element of the semigroup such that $m > 2 - 2g$ where, now, g is the genus of semigroup.

Classical Feng Rao distance

- Called δ, was introduced in 1993 for Feng and Rao.
- Feng Rao distance gave a better bound for the distance than the Goppa bound:

$$d \geq \delta(m+1) \geq m+2-2g \text{ when } m > 2g-2.$$

- For $m > 2c-1$ we have $\delta(m) = m+1-2g$.

Definition

[1993] (Adapted to semigroups)

- Given a semigroup $S = \{\rho_0 = 0 < \rho_1 < \ldots < \rho_n < \ldots\}$,
- We define $\nu(\rho_l) = \sharp\{(\rho_i, \rho_j) \in S^2 : \rho_i + \rho_j = \rho_l\}$,
- The Feng-Rao distance for $m \in S$ is defined as

$$d(m) = min\{\nu(\rho_l) : \rho_l \geq m\}$$

Using divisors

- This definition can be translate in terms of divisors:
- For S a numerical semigroup and $m_1 \in S$, let

$$\mathrm{D}(m_1) := \{p \in S \mid m_1 - p \in S\}$$

and

$$\nu(m_1) := \sharp \mathrm{D}(m_1).$$

- The (classical) Feng-Rao distance of S is defined by:

$$\delta_{FR}(m) := \min\{\nu(m_1) \mid m_1 \geq m, \quad m_1 \in S\}.$$

Generalized distances

Historical hits

- Hamming distances can be generalized from vectorial subspaces of dimension one to higher dimension. Instead of comparing two elements we to compare r elements (or basis of spaces of dimension r). Classically, this is called “weights hierarchy”
  1. Introduced in 1977 for Helleseth, Klove and Mykkleveit,
  2. and in 1991 for Wei.
- The bounds for the general setting were obtain in two steps:
  1. In 1998, Heijnen and Pellikaan, proved that Hamming distances are bounded for generalized Feng Rao distances.
  2. In 2003, Farrán and Munuera, proved that Feng Rao generalized distances are better than the generalized Goppa bound.

Generalized distances

Historical hits

In this last paper the authors proved too, that the difference between FR generalized distances and Goppa bound depends only on r, the dimension, and S, the semigroup, not on m if $m \geq 2c - 1$.

$$\delta^r_{FR}(m) = m + 1 - 2g + E(r, S) \tag{1}$$

The $E(r, S)$ are called Feng Rao numbers.

Computing FR-numbers

To compute this generalized FR distances, or equivalently, the Feng Rao numbers is a very complex problem compared with the classical distance. There are several papers giving formulas for different families of numerical semigroups in the classical case.

- For generalized FR distances.
  1. In 2000, Barbero and Munuera compute generalized weights for hemitian codes (where $S = \langle q, q+1 \rangle$), with q the power of a prime.
  2. In 2003, Farrán and Munuera compute Feng Rao generalized distance for $r = 2$ and semigroups generated by two elements ($E_2 = \rho_2$), working with “desserts”.
  3. In 2010, Farrán, García-Sánchez and Llena compute FR numbers for every r and $S = \langle a, a+1 \rangle$ proving that $E_r = \rho_r$, and we compute too the generalized FR distance for $S = \langle 2, 2g+1 \rangle$.
  4. In 2011, Delgado, Farrán, García-Sánchez and Llena, compute for every r and $S = \langle a, a+1, \ldots, a+h \rangle$ the FR distances. This case, is not a symmetric semigroup and we obtain better bound than ρ_r for some r.
  5. And we just working in the computations for $S = \langle a, b \rangle$ to prove, in this case, that the bound is achieved i.e. $E(r, S) = \rho_r$.

Generalized FR distances using divisors

- Definitions
  1. Let S be a numerical semigroup. For $m_1, \dots, m_r \in S$, let

$$\begin{aligned} \mathrm{D}(m_1, \dots, m_r) &= \mathrm{D}(m_1) \cup \dots \cup \mathrm{D}(m_r) \\ &= \{p \in S \mid m_i - p \in S, \quad \exists i \in \{1, \dots, r\}\} \end{aligned}$$

     and $\nu(m_1, \dots, m_r) := \sharp \mathrm{D}(m_1, \dots, m_r)$.
  2. For any integer $r \geq 1$, the r-th Feng-Rao distance of S is defined by:

$$\delta_{FR}^r(m) := \min\{\nu(m_1, \dots, m_r) \mid m \leq m_1 < \dots < m_r, \quad m_i \in S\}.$$

Generalized FR distances using divisors

- Ideas
  1. The principal idea to compute generalized FR distances is:
     - If $s \in S$ then $\mathrm{D}(m) \subset \mathrm{D}(m+s)$, so
     $$\mathrm{D}(m, m+s) = \mathrm{D}(m+s)$$
     This indicates which is the way to choose elements to obtain the minimal distance.
  2. We take $\mathrm{m} = 2c - 1$. And try to obtain $\{m_2, \dots, m_r\}$ such that $\sharp\mathrm{D}(\mathrm{m}, m_2, \dots, m_r)$ is minimal.

Generalized FR distances using divisors:Case $\langle a, a+1 \rangle$

We are helped for this type of picture, $S = \langle 11, 12 \rangle$

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

Where red values are $\mathrm{D(m)}$. Note that it's necessary to require that all red elements are in S but this is true because $\mathrm{m} \geq 2c - 1$.

$S = \langle 11, 12\rangle$

Now we need to choose black element bigger than m such that the divisors added are the minimum possible. We left divisors of $m+2$ (in green)...

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

$S = \langle 11, 12\rangle$

... and $m + 28$(in blue).

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

We can see that

$$D(m, m+28) = D(m, m+4, m+5, m+6, m+16, m+17, m+28).$$

Some useful definitions and ideas

From this we can divide “black elements” in three disjoint subset.

1. The Tail $T(\mathrm{m}) = \{s \in S : s \notin \mathrm{D}(\mathrm{m}), s < \mathrm{m}\}$.
2. The Ground $G(\mathrm{m}) = \{s \in S : \mathrm{m} < s < \mathrm{m} + a + 1\}$.
3. The Head $H(\mathrm{m}) = \{s \in S : \mathrm{m} + a + 1 \geq s\}$.

1. To compute FR numbers we can not add to $\mathrm{D}(\mathrm{m})$ elements of its Tail.
2. Elements in the Ground add elements in the Tail.
3. If we have two consecutive (or more) elements in the Ground, we can take his head to add one (or various) new elements, i.e. If we consider $\mathrm{D}(\mathrm{m}, \mathrm{m}+4, \mathrm{m}+5)$ then we have that $\mathrm{D}(\mathrm{m}, \mathrm{m}+4, \mathrm{m}+5, \mathrm{m}+16) = \mathrm{D}(\mathrm{m}, \mathrm{m}+4, \mathrm{m}+5) \cup \{\mathrm{m}+16\}$.
4. If we control the divisor of element of Ground, we control the tail.
5. Amenable sets are those $M \subset [m, \infty)$ which $D(M) \cap [\mathrm{m}, \infty) = M$.

Amenable and non-amenable sets

$S = \langle 11, 12 \rangle$

In blue amenable set, in red non amenable.

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11

$S = \langle 11, 12 \rangle$

Another amenable set in blue.

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11

Choosing

$S = \langle 11, 12\rangle$

So for $r = 2$ we can choose $\mathrm{D}(\mathrm{m}, \mathrm{m}+1)$ or $\mathrm{D}(\mathrm{m}, \mathrm{m}+a)$ both of them are minimal with respect to add the minimum number of divisors at $\mathrm{D}(\mathrm{m})$. They add 11 each one.

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

But for $r = 3$ we have only a election $\mathrm{D}(\mathrm{m}, \mathrm{m}+a+1, \mathrm{m}+1)$, cause $\mathrm{m}+a+1$ is the head form $\mathrm{m}, \mathrm{m}+1$.

proof

And we finally proved that to compute

$$\mathrm{E}(r, \langle a, a+1\rangle) = \rho_r$$

- It's enough to take $\mathrm{D}(\mathrm{m}, \mathrm{m}+\rho_r-\rho_{r-1}, \mathrm{m}+\rho_r-\rho_{r-2}, \dots, \mathrm{m}+\rho_r-\rho_1)$.
- In general, for each r we define $b(r)$ as the unique natural number such that $\frac{1}{2}b(r)(b(r)+1) < r \leq \frac{1}{2}(b(r)+1)(b(r)+2)$.
- Then $b(r)$ is the length of the ground and must be contain m, then we put the other elements on the top while holes.

Example

$S = \langle 11, 12\rangle$

For $r = 4$ we have $b(r) = 2$ (m and two more elements in the ground). Ground in green, Tail in yellow, possible divisors out of the ground in blue (we must choose one of them)

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

Other elections

$S = \langle 11, 12\rangle$

But we can choose too (in this case we have only one in blue):

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

more elections

$S = \langle 11, 12 \rangle$

even (having, again, one element in blue. We can't choose $\mathrm{m}+23$)

m+24	m+25	m+26	m+27	m+28	m+29	m+30	m+31	m+32	m+33	m+34	m+35
m+12	m+13	m+14	m+15	m+16	m+17	m+18	m+19	m+20	m+21	m+22	m+23
m	m+1	m+2	m+3	m+4	m+5	m+6	m+7	m+8	m+9	m+10	m+11
m-12	m-11	m-10	m-9	m-8	m-7	m-6	m-5	m-4	m-3	m-2	m-1
m-24	m-23	m-22	m-21	m-20	m-19	m-18	m-17	m-16	m-15	m-14	m-13
m-36	m-35	m-34	m-33	m-32	m-31	m-30	m-29	m-28	m-27	m-26	m-25
m-48	m-47	m-46	m-45	m-44	m-43	m-42	m-41	m-40	m-39	m-38	m-37
m-60	m-59	m-58	m-57	m-56	m-55	m-54	m-53	m-52	m-51	m-50	m-49
m-72	m-71	m-70	m-69	m-68	m-67	m-66	m-65	m-64	m-63	m-62	m-61
m-84	m-83	m-82	m-81	m-80	m-79	m-78	m-77	m-76	m-75	m-74	m-73
m-96	m-95	m-94	m-93	m-92	m-91	m-90	m-89	m-88	m-87	m-86	m-85
m-108	m-107	m-106	m-105	m-104	m-103	m-102	m-101	m-100	m-99	m-98	m-97
m-120	m-119	m-118	m-117	m-116	m-115	m-114	m-113	m-112	m-111	m-110	m-109
m-132	m-131	m-130	m-129	m-128	m-127	m-126	m-125	m-124	m-123	m-122	m-121

In this case we have a formula to compute $\sharp(\mathrm{D}(\mathrm{m}, \mathrm{m}+i_1, \ldots \mathrm{m}+i_t) \setminus \mathrm{D}(\mathrm{m})) = \sum_{j=1}^{t}(i_j - i_{j-1})(a+1-i_j)$ taking $i_0 = 0$ and $i < a$.

is a differentiable formula, and it's very easy to obtain the minimum.

In the two following cases we have non continuum formulas.

Case $\langle a, a+1, \dots, a+h \rangle$

In this case the formulas are:

$$\begin{aligned} \sharp \mathrm{D}(\mathrm{m}, \mathrm{m}+i_1, \dots, \mathrm{m}+i_t) &= \sharp \mathrm{D}(\mathrm{m}) + \sum_{k=1}^{t} \sum_{j=i_{k-1}+1}^{i_k} \left\lceil \frac{a+b-j}{b} \right\rceil - \left\lceil \frac{i_k - j}{b} \right\rceil \\ &= \sharp \mathrm{D}(\mathrm{m}) + \sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{k=1}^{t} \sum_{j=i_{k-1}+1}^{i_k} \left\lceil \frac{i_k - j}{b} \right\rceil . \end{aligned}$$

Ceil functions are not differentiable.

Case $\langle a, a+1, \ldots a+h \rangle$

Let $S = \langle 19, 20, 21, 22, 23 \rangle$ be a generated interval semigroup.
In red $\mathrm{D(m)}$ in blue $\mathrm{D(m+49, m+50)}$. By space question we are removed m in all hole.

+69	+70	+71	+72	+73	+74	+75	+76	+77	78	+79	+80	+81	+82	+83	+84	+85	+86	+87	88	+89	+90	+91
+46	+47	+48	+49	+50	+51	+52	+53	+54	+55	+56	+57	58	+59	+60	+61	+62	+63	+64	+65	+66	+67	+68
+23	+24	+25	+26	+27	+28	+29	+30	+31	+32	+33	+34	+35	+36	+37	38	+39	+40	+41	+42	+43	+44	+45
m	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	+17	+18	+19	+20	+21	+22
-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
-46	-45	-44	-43	-42	-41	-40	-39	-38	-37	-36	-35	-34	-33	-32	-31	-30	-29	-28	-27	-26	-25	-24
-69	-68	-67	-66	-65	-64	-63	-62	-61	-60	-59	-58	-57	-56	-55	-54	-53	-52	-51	-50	-49	-48	-47
-92	-91	-90	-89	-88	-87	-86	-85	-84	-83	-82	-81	-80	-79	-78	-77	-76	-75	-74	-73	-72	-71	-70

We can observe that $+72$ can still be added because $+72 - 19, +72 - 20, +72 - 21$ are divisors and not in the set choose.
If we want to add a new element we necessarily must be add $+2$, then $+25$ and the $+48$.
We can make a proof using moving to the left just taking one away or adding one to move to the right. And removing the bigger element in a amenable configurations.

generated by two

In this case we have too a non continuum formula. Let $0 < i_1 < \cdots < i_{k+1} < b$. Then

$$\sharp \mathrm{D}(\mathrm{m}, \mathrm{m} \oplus i_1, \ldots, \mathrm{m} \oplus i_{k+1}) = \mathrm{m} + 1 - 2g + \sum_{j=0}^{k} (i_{j+1} - i_j) \left(a - \left\lfloor \frac{i_{j+1} a}{b} \right\rfloor \right),$$

where i_0 is taken to be 0.

And to obtain an amenable configuration we can to put the elements in a new order, as we can see in the picture. $S = \langle 5, 23\rangle$. The ground in red.

203	208	213	218	223	228	233	238	243	248	253	258	263	268	273	278	283	288	293	298	303	308	313
180	185	190	195	200	205	210	215	220	225	230	235	240	245	250	255	260	265	270	275	280	285	290
157	162	167	172	177	182	187	192	197	202	207	212	217	222	227	232	237	242	247	252	257	262	267
134	139	144	149	154	159	164	169	174	179	184	189	194	199	204	209	214	219	224	229	234	239	244
111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186	191	196	201	206	211	216	221
88	93	98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193	198
65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	m

We can observe that there are different length for each piece of ground (4 o 5 in our case). But in this case we can control the divisors “easier” than above cases. In blue divisors of 242 over the ground in green divisors of 242 under the ground, (divisors that not are divisors of m). In yellow divisors of m.

thanks !!