On the Möbius function for arithmetic numerical semigroups

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Outline

Poset structure induced by NS on $\mathbb{Z}$

Let $S = \langle a_1, a_2, \ldots, a_n \rangle$ be the numerical semigroup generated by the relatively prime positive integers $a_1, a_2, \ldots, a_n$, that is,

$$S = \langle a_1, a_2, \ldots, a_n \rangle = \left\{ m_1 a_1 + m_2 a_2 + \cdots + m_n a_n \mid m_1, \ldots, m_n \in \mathbb{N} \right\}.$$

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Let S be a NS. We consider the poset $(\mathbb{Z}, <_S)$ defined by

$$x <_S y \quad \Longleftrightarrow \quad y - x \in S \setminus \{0\},$$

for all $x, y \in \mathbb{Z}$.

Chains in $(\mathbb{Z}, <_S)$

A chain of length $p \geq 0$ between x and y is a finite sequence $(x_0, x_1, \ldots, x_p)$ of $p+1$ elements such that

$$x = x_0 <_S x_1 <_S \cdots <_S x_p = y.$$

Let $c_p(x, y)$ denote the number of chains of length p between x and y.

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Recursive definition

- $c_0(x, y) = \begin{cases} 0 \text{ if } x \neq y, \\ 1 \text{ if } x = y, \end{cases}$
- $c_p(x, y) = 0$, if $x \not<_S y$, for $p \geq 1$,
- $c_p(x, y) = \sum_{x <_S x' \leq_S y} c_{p-1}(x', y) = \sum_{x \leq_S y' <_S y} c_{p-1}(x, y')$, if $x <_S y$, for $p \geq 1$.

The Möbius function of $(\mathbb{Z}, <_S)$

Let $\mu_S : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ be the Möbius function associated whith $(\mathbb{Z}, <_S)$, that is,

$$\mu_S(x, y) = \sum_{p \geq 0} (-1)^p c_p(x, y),$$

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- Introduced in 1964 by Rota for all posets.
- Generalization of the classical Möbius function $\mu(n)$, which corresponds to the Möbius function of the set of all positive integers partially ordered by divisibility.
- Möbius inversion formula : for $\mathcal{P}$ a poset, given a function $f : \mathcal{P} \longrightarrow \mathbb{R}$ with finite support, the value $f(m)$ can be recovered from the function $g(x) = \sum_{y \leq x} f(y)$ using Möbius function $f(m) = \sum_{x \leq m} \mu_{\mathcal{P}}(x, m) g(x)$.

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In the sequel, we only consider $\mu_S : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by

$$\mu_S(x) = \begin{cases} \mu_S(0,x) & , \text{ if } x \in S, \\ 0 & , \text{ otherwise.} \end{cases}$$

The Möbius function of $(\mathbb{Z}, <_S)$

For every $x \in S \setminus \{0\}$,

$$\begin{aligned}
\mu_S(x) &= \mu_S(0,x) = \sum_{p\geq 0} (-1)^p c_p(0,x) \\
&= \underbrace{c_0(0,x)}_{0} + \sum_{p\geq 1} (-1)^p \sum_{0\leq_S x' <_S x} c_{p-1}(0,x') \\
&= - \sum_{0\leq_S x' <_S x} \sum_{p\geq 0} (-1)^p c_p(0,x') \\
&= - \sum_{0\leq_S x' <_S x} \mu_S(0,x') = - \sum_{0\leq_S x' <_S x} \mu_S(x')
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&= -\sum_{0\leq_S x' <_S x} \sum_{p\geq 0} (-1)^p c_p(0,x') \\
&= -\sum_{0\leq_S x' <_S x} \mu_S(0,x') = -\sum_{0\leq_S x' <_S x} \mu_S(x')
\end{aligned}$$

Proposition

$$\mu_S(x) = -\sum_{y\in S\setminus\{0\}} \mu_S(x-y) \quad \Longleftrightarrow \quad \sum_{y\in S} \mu_S(x-y) = 0, \quad \forall x \in \mathbb{Z} \setminus \{0\}.$$

For semigroups with two generators

Theorem (Deddens,1979)

Let $S = \langle a, b \rangle$ where a and b are relatively prime positive integers. Then, for all $x \in \mathbb{N}$,

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \equiv 0 \text{ or } a+b \pmod{ab}, \\ -1 & \text{if } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

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Proof given by Deddens

A recursive case-by-case analysis.

A new direct proof of Deddens' result

Let $x \in S = \langle a, b \rangle = \{m_a a + m_b b \mid m_a, m_b \in \mathbb{N}\}$. For $x \neq 0$,

$$\mu_S(x) = -\sum_{y \in S \setminus \{0\}} \mu_S(x-y) = -\sum_{\substack{y \in S \setminus \{0\} \\ y-a \in S}} \mu_S(x-y) - \sum_{\substack{y \in S \setminus \{0\} \\ y-a \notin S}} \mu_S(x-y).$$

Since

$$\sum_{\substack{y \in S \setminus \{0\} \\ y-a \in S}} \mu_S(x-y) = \sum_{z \in S} \mu_S((x-a)-z) = 0 \quad , \text{ for } x-a \neq 0,$$

we have

$$\mu_S(x) = -\sum_{\substack{y \in S \setminus \{0\} \\ y-a \notin S}} \mu_S(x-y), \quad \text{for } x \in S \setminus \{0, a\}.$$

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$$\mu_S(x) = -\sum_{\substack{y\in S\setminus\{0\}\\ y-a\notin S}} \mu_S(x-y), \quad \text{for } x \in S\setminus\{0,a\}.$$

Moreover,

$$\{y\in S\setminus\{0\} \mid y-a\notin S\} = \{m_b b \mid m_b \in \{1,2,\ldots,a-1\}\}.$$

Thus,

$$\mu_S(x) = -\sum_{m_b=1}^{a-1} \mu_S(x-m_b b), \quad \text{for } x \in S\setminus\{0,a\}.$$

Finally,

$$\mu_S(x-b) = -\sum_{m_b=2}^{a} \mu_S(x-m_b b), \quad \text{for } x-b \in S\setminus\{0,a\}.$$

A new direct proof of Deddens' result

$$\mu_S(x) = -\sum_{m_b=1}^{a-1} \mu_S(x - m_b b), \quad \text{for } x \in S \setminus \{0, a\}.$$

$$\mu_S(x - b) = -\sum_{m_b=2}^{a} \mu_S(x - m_b b), \quad \text{for } x - b \in S \setminus \{0, a\}.$$

For $x \in S \setminus \{0, a, b, a + b\}$, we obtain that

$$\begin{aligned}
\mu_S(x) &= -\sum_{m_b=1}^{a-1} \mu_S(x - m_b b) = -\mu_S(x - b) - \sum_{m_b=2}^{a-1} \mu_S(x - m_b b) \\
&= \sum_{m_b=2}^{a} \mu_S(x - m_b b) - \sum_{m_b=2}^{a-1} \mu_S(x - m_b b) \\
&= \mu_S(x - ab)
\end{aligned}$$

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For every $x \in \mathbb{Z} \setminus \{0, a, b, a+b\}$,

$$\mu_S(x) = \mu_S(x - ab).$$

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Since

- $\mu_S(x) = 0$, for all $x < 0$,
- $\mu_S(0) = 1$,
- $\mu_S(a) = \mu_S(b) = -1$,
- $\mu_S(a+b) = 2 - 1 = 1$,

the formula of Deddens follows:

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } a+b \pmod{ab}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

Arithmetic Numerical Semigroups

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For $a, d \in \mathbb{N}$ such that $\gcd(a, d) = 1$ and $k \in \{1, 2, \ldots, a-1\}$, we consider the arithmetic numerical semigroup

$$\begin{aligned} S &= \langle a, a+d, \ldots, a+kd \rangle \\ &= \{x_0 a + x_1(a+d) + \cdots + x_k(a+kd) \mid x_0, x_1, \ldots, x_k \in \mathbb{N}\}. \end{aligned}$$

A unique representation in ANS

Proposition

Let $x \in S = \langle a, a+d, \ldots, a+kd \rangle$. Let $a = qk + r$ where $r \in \{0, 1, \ldots, k-1\}$. Then, there exists a unique triplet of elements $(x_0, i, x_k) \in \mathbb{N} \times \{0, 1, \ldots, k-1\} \times \{0, 1, \ldots, q\}$ such that

$$x = x_0 a + \lceil i/k \rceil (a + id) + x_k (a + kd),$$

where $i + x_k k < a$. We denote by $x[x_0, i, x_k]$ this representation.

- if $0 \leq i_1 + i_2 \leq k$,

$$(a + i_1 d) + (a + i_2 d) = a + (a + (i_1 + i_2)d),$$

- if $k \leq i_1 + i_2 \leq 2k$,

$$(a + i_1 d) + (a + i_2 d) = (a + (i_1 + i_2 - k)d) + (a + kd),$$

- $q(a + kd) + (a + rd) = (q + d + 1)a.$

A valuable tool in ANS : the Apéry set

The Apéry set of S with respect with $m \in S$ is defined as

$$Ap(S; m) = \{x \in S \mid x - m \notin S\}.$$

The Apéry set $Ap(S; m)$ constitutes a complete set a of residues $\bmod m$.

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Theorem (Roberts,1956)

Let $S = \langle a, a+d, \ldots, a+kd \rangle$ with $\gcd(a,d) = 1$, the Apéry set of S is

$$Ap(S;a) = \left\{ \left\lceil \frac{i}{k} \right\rceil a + id \;\middle|\; i \in \{0,1,\ldots,a-1\} \right\}.$$

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Let $S = \langle a, a+d, \ldots, a+kd \rangle$ with $\gcd(a,d) = 1$, the Apéry set of S is

$$Ap(S;a) = \left\{ \left\lceil \frac{i}{k} \right\rceil a + id \;\middle|\; i \in \{0, 1, \ldots, a-1\} \right\}.$$

Let $x = m_a a + m_d d$ such that $m_a \geq 0$ and $0 \leq m_d \leq a - 1$. Then,

$$x \in S \iff m_a \geq \left\lceil \frac{m_d}{k} \right\rceil.$$

A recursive identity for μ_S of ANS

Let $S = \langle a, a+d, \ldots, a+kd \rangle$, where $\gcd(a,d) = 1$.
Let $x \in S \setminus \{0\}$. As for NS with 2 generators, we already know that

$$\begin{aligned}
\mu_S(x) &= -\sum_{y \in S \setminus \{0\}} \mu_S(x-y) \\
&= -\sum_{\substack{y \in S \setminus \{0\} \\ y-a \in S}} \mu_S(x-y) - \sum_{\substack{y \in S \setminus \{0\} \\ y-a \notin S}} \mu_S(x-y) \\
&= -\sum_{z \in S} \mu_S((x-a)-z) - \sum_{\substack{y \in S \setminus \{0\} \\ y-a \notin S}} \mu_S(x-y) \\
&= -\sum_{\substack{y \in S \setminus \{0\} \\ y-a \notin S}} \mu_S(x-y) \quad \text{if} \quad x-a \neq 0.
\end{aligned}$$

Now, we determine the set $\{y \in S \setminus \{0\} \mid y-a \notin S\}$.

A recursive identity for μ_S of ANS

First, we have

$$\{y \in S \setminus \{0\} \mid y - a \notin S\} = Ap(S;a) \setminus \{0\}.$$

Let $a = qk + r$ be the Euclidean division of a by k.

- $\underline{r = 0}$:

$$\begin{aligned} Ap(S,a)\setminus\{0\} = \ & \{y_k(a+kd) \mid y_k \in \{1,\ldots,q-1\}\} \\ & \bigcup \left\{(a+id) + y_k(a+kd) \;\middle|\; \begin{array}{l} y_k \in \{0,\ldots,q-1\} \\ i \in \{1,\ldots,k-1\} \end{array} \right\} \end{aligned}$$

- $\underline{r \geq 1}$:

$$\begin{aligned} Ap(S,a)\setminus\{0\} = \ & \{y_k(a+kd) \mid y_k \in \{1,\ldots,q\}\} \\ & \bigcup \left\{(a+id) + y_k(a+kd) \;\middle|\; \begin{array}{l} y_k \in \{0,\ldots,q\} \\ i \in \{1,\ldots,r-1\} \end{array} \right\} \\ & \bigcup \left\{(a+id) + y_k(a+kd) \;\middle|\; \begin{array}{l} y_k \in \{0,\ldots,q-1\} \\ i \in \{r,\ldots,k-1\} \end{array} \right\} \end{aligned}$$

A recursive identity for μ_S of ANS

Suppose that $r = 0$, i.e. $a = qk$. For $x \in S \setminus \{0, a\}$,

$$\mu_S(x) = -\sum_{y_k=1}^{q-1} \mu_S(x - y_k(a + kd)) - \sum_{i=1}^{k-1} \sum_{y_k=0}^{q-1} \mu_S(x - (a + id) - y_k(a + kd)).$$

Applying this formula for $x - (a + kd) \in S \setminus \{0, a\}$, we obtain that

$$\mu_S(x) = \mu_S(x - q(a + kd)) + \sum_{i=1}^{k-1} \mu_S(x - (a + id) - q(a + kd)) - \mu_S(x - (a + id))$$

for every $x \in S \setminus \{0, a, a + kd, a + (a + kd)\}$.

The technique is similar in the case $r \geq 1$.

A recursive identity for μ_S of ANS

Theorem (C., Ramírez Alfonsín)

Let $S = \langle a, a+d, \ldots, a+kd \rangle$, where $\gcd(a,d) = 1$.
Let $x \in S \setminus \{0, a, a+kd, a+(a+kd)\}$. If $r = 0$, then

$$\mu_S(x) = \mu_S(x-q(a+kd)) + \sum_{i=1}^{k-1} \mu_S(x-(a+id)-q(a+kd)) - \mu_S(x-(a+id)).$$

If $r \geq 1$, then

$$\mu_S(x) = \mu_S(x-(q+1)(a+kd)) + \sum_{i=1}^{r-1} \mu_S(x-(a+id)-(q+1)(a+kd))$$
$$\sum_{i=r}^{k-1} \mu_S(x-(a+id)-q(a+kd)) - \sum_{i=1}^{k-1} \mu_S(x-(a+id)).$$

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Now $k = 2$ and $r = 0$, i.e. $S = \langle 2q, 2q+d, 2q+2d \rangle$ with $\gcd(q, d) = 1$. The previous theorem becomes

$$\mu_S(x) = \mu_S(x - q(a+2d)) + \mu_S(x - (a+d) - q(a+2d)) - \mu_S(x - (a+d)).$$

Moreover, with the unique representation

$$x[x_0, i, x_2] = \begin{cases} x_0 a + x_2(a+2d) & \text{if } i = 0 \text{ and } 0 \le x_2 \le q-1, \\ x_0 a + (a+d) + x_2(a+2d) & \text{if } i = 1 \text{ and } 0 \le x_2 \le q-1, \end{cases}$$

and the equality

$$q(a+2d) = (q+d)a,$$

we obtain the following refinement.

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Recursive identities

Let $x_0, i, x_2 \in \mathbb{N}$ such that $i \in \{0,1\}$ and $i + 2x_2 < a$. Then,

$$\begin{aligned}\mu_S\left(x[x_0, i, 0]\right) = \ & \mu_S\left(x[x_0 - (q+d), i, 0]\right) \\ & + \mu_S\left(x[x_0 - (q+d) - 1, i, q-1]\right) \\ & - \mu_S\left(x[x_0 - 2(q+d) - 1, i, q-1]\right)\end{aligned}$$

and

$$\begin{aligned}\mu_S\left(x[x_0, i, x_2]\right) = \ & \mu_S\left(x[x_0 - (q+d), i, x_2]\right) \\ & + \mu_S\left(x[x_0 - 1, i, x_2 - 1]\right) \\ & - \mu_S\left(x[x_0 - (q+d) - 1, i, x_2 - 1]\right)\end{aligned}$$

in the case where $x_2 \geq 1$.

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
0	1	−1	0	0	0	0	0	0	0	0	0
1	−1	2	−1	0	0	0	0	0	0	0	0
2	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
	0	0	0	−1	2	−1	0	0	0	0	0
	0	0	0	0	−1	2	−1	0	0	0	0
	0	0	0	0	0	−1	2	−1	0	0	0
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$q-1$	0	0	0	0	0	0	0	0	0	−1	2
q	0	0	0	0	0	0	0	0	0	0	−1
$q+1$	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
$q+d-1$	0	0	0	0	0	0	0	0	0	0	0

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2	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
	0	0	0	−1	2	−1	0	0	0	0	0
	0	0	0	0	−1	2	−1	0	0	0	0
	0	0	0	0	0	−1	2	−1	0	0	0
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$q-1$	0	0	0	0	0	0	0	0	0	−1	2
q	0	0	0	0	0	0	0	0	0	0	−1
$q+1$	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
$q+d-1$	0	0	0	0	0	0	0	0	0	0	0

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$q+d+2$	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
	0	0	0	−1	2	−1	0	0	0	0	0
	0	0	0	0	−1	2	−1	0	0	0	0
	0	0	0	0	0	−1	2	−1	0	0	0
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$2q+d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$2q+d$	2	−1	0	0	0	0	0	0	0	0	−1
$2q+d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$2q+2d-1$	0	0	0	−1	2	−1	0	0	0	0	0

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$q+d+2$	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
	0	0	0	−1	2	−1	0	0	0	0	0
	0	0	0	0	−1	2	−1	0	0	0	0
	0	0	0	0	0	−1	2	−1	0	0	0
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$2q+d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$2q+d$	2	−1	0	0	0	0	0	0	0	0	−1
$2q+d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$2q+2d-1$	0	0	0	−1	2	−1	0	0	0	0	0

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$q+d$	1	−1	0	0	0	0	0	0	0	0	0
$q+d+1$	−1	2	−1	0	0	0	0	0	0	0	0
$q+d+2$	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
	0	0	0	−1	2	−1	0	0	0	0	0
	0	0	0	0	−1	2	−1	0	0	0	0
	0	0	0	0	0	−1	2	−1	0	0	0
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$2q+d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$2q+d$	2	−1	0	0	0	0	0	0	0	0	−1
$2q+d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$2q+2d-1$	0	0	0	−1	2	−1	0	0	0	0	0

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$2q+2d$	1	−1	0	0	−1	2	−1	0	0	0	0
$2q+2d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$2q+2d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
	0	0	0	0	−1	2	−1	0	0	−1	2
	0	0	0	0	0	−1	2	−1	0	0	−1
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$3q+2d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$3q+2d$	2	−1	0	0	0	0	0	0	0	0	−1
$3q+2d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$3q+3d-1$	0	0	0	−1	2	−1	0	0	0	0	0

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$2q+2d$	1	−1	0	0	−1	2	−1	0	0	0	0
$2q+2d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$2q+2d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
	0	0	0	0	−1	2	−1	0	0	−1	2
	0	0	0	0	0	−1	2	−1	0	0	−1
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$3q+2d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$3q+2d$	2	−1	0	0	0	0	0	0	0	0	−1
$3q+2d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$3q+3d-1$	0	0	0	−1	2	−1	0	0	0	0	0

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$2q+2d$	1	−1	0	0	−1	2	−1	0	0	0	0
$2q+2d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$2q+2d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
	0	0	0	0	−1	2	−1	0	0	−1	2
	0	0	0	0	0	−1	2	−1	0	0	−1
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$3q+2d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$3q+2d$	2	−1	0	0	0	0	0	0	0	0	−1
$3q+2d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$3q+3d-1$	0	0	0	−1	2	−1	0	0	0	0	0

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$3q+3d$	1	−1	0	0	−1	2	−1	0	0	0	0
$3q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$3q+3d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$4q+2d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$4q+2d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$4q+2d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$4q+3d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$4q+3d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$4q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$4q+4d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$3q+3d$	1	−1	0	0	−1	2	−1	0	0	0	0
$3q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$3q+3d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$4q+2d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$4q+2d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$4q+2d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$4q+3d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$4q+3d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$4q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$4q+4d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$3q+3d$	1	−1	0	0	−1	2	−1	0	0	0	0
$3q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$3q+3d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$4q+2d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$4q+2d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$4q+2d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$4q+3d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$4q+3d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$4q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$4q+4d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$3q+3d$	1	−1	0	0	−1	2	−1	0	0	0	0
$3q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$3q+3d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$4q+2d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$4q+2d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$4q+2d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$4q+3d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$4q+3d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$4q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$4q+4d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$4q+4d$	1	−1	0	0	−1	2	−1	0	0	−1	2
$4q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
$4q+4d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$5q+3d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$5q+3d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$5q+3d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$5q+4d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$5q+4d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$5q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$5q+5d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$4q+4d$	1	−1	0	0	−1	2	−1	0	0	−1	2
$4q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
$4q+4d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$5q+3d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$5q+3d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$5q+3d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$5q+4d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$5q+4d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$5q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$5q+5d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$4q+4d$	1	−1	0	0	−1	2	−1	0	0	−1	2
$4q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
$4q+4d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$5q+3d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$5q+3d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$5q+3d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$5q+4d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$5q+4d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$5q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$5q+5d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$4q+4d$	1	−1	0	0	−1	2	−1	0	0	−1	2
$4q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
$4q+4d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
$5q+3d-1$	−1	0	0	0	−1	2	−1	0	0	−1	2
$5q+3d$	2	−1	0	0	0	−1	2	−1	0	0	−1
$5q+3d+1$	−1	2	−1	0	0	0	−1	2	−1	0	0
	0	−1	2	−1	0	0	0	−1	2	−1	0
	0	0	−1	2	−1	0	0	0	−1	2	−1
$5q+4d-1$	−1	0	0	−1	2	−1	0	0	0	−1	2
$5q+4d$	2	−1	0	0	−1	2	−1	0	0	0	−1
$5q+4d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$5q+5d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$5q+5d$	0	−1	0	0	−1	2	−1	0	0	−1	2
$5q+5d+1$	1	1	−1	0	0	−1	2	−1	0	0	−1
$5q+5d+2$	−1	1	1	−1	0	0	−1	2	−1	0	0
	0	−1	1	1	−1	0	0	−1	2	−1	0
	0	0	−1	1	1	−1	0	0	−1	2	−1
$6q+4d-1$	−1	0	0	−1	1	1	−1	0	0	−1	2
$6q+4d$	2	−1	0	0	−1	1	1	−1	0	0	−1
$6q+4d+1$	−1	2	−1	0	0	−1	1	1	−1	0	0
	0	−1	2	−1	0	0	−1	1	1	−1	0
	0	0	−1	2	−1	0	0	−1	1	1	−1
$6q+5d-1$	−1	0	0	−1	2	−1	0	0	−1	1	1
$6q+5d$	2	−1	0	0	−1	2	−1	0	0	−1	1
$6q+5d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$6q+6d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

$5q+5d = 6q+3d-1, \quad 5q+5d+1 = 6q+3d, \quad 5q+5d+2 = 6q+3d+1$

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$5q+5d$	0	−1	0	0	−1	2	−1	0	0	−1	2
$5q+5d+1$	1	1	−1	0	0	−1	2	−1	0	0	−1
$5q+5d+2$	−1	1	1	−1	0	0	−1	2	−1	0	0
	0	−1	1	1	−1	0	0	−1	2	−1	0
	0	0	−1	1	1	−1	0	0	−1	2	−1
$6q+4d-1$	−1	0	0	−1	1	1	−1	0	0	−1	2
$6q+4d$	2	−1	0	0	−1	1	1	−1	0	0	−1
$6q+4d+1$	−1	2	−1	0	0	−1	1	1	−1	0	0
	0	−1	2	−1	0	0	−1	1	1	−1	0
	0	0	−1	2	−1	0	0	−1	1	1	−1
$6q+5d-1$	−1	0	0	−1	2	−1	0	0	−1	1	1
$6q+5d$	2	−1	0	0	−1	2	−1	0	0	−1	1
$6q+5d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$6q+6d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

$5q+5d = 6q+3d-1, \quad 5q+5d+1 = 6q+3d, \quad 5q+5d+2 = 6q+3d+1$

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$5q+5d$	0	−1	0	0	−1	2	−1	0	0	−1	2
$5q+5d+1$	1	1	−1	0	0	−1	2	−1	0	0	−1
$5q+5d+2$	−1	1	1	−1	0	0	−1	2	−1	0	0
	0	−1	1	1	−1	0	0	−1	2	−1	0
	0	0	−1	1	1	−1	0	0	−1	2	−1
$6q+4d-1$	−1	0	0	−1	1	1	−1	0	0	−1	2
$6q+4d$	2	−1	0	0	−1	1	1	−1	0	0	−1
$6q+4d+1$	−1	2	−1	0	0	−1	1	1	−1	0	0
	0	−1	2	−1	0	0	−1	1	1	−1	0
	0	0	−1	2	−1	0	0	−1	1	1	−1
$6q+5d-1$	−1	0	0	−1	2	−1	0	0	−1	1	1
$6q+5d$	2	−1	0	0	−1	2	−1	0	0	−1	1
$6q+5d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$6q+6d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

$5q+5d = 6q+3d-1$, $\quad 5q+5d+1 = 6q+3d$, $\quad 5q+5d+2 = 6q+3d+1$

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Example for $q = 11$ and $d = 5$: the first values of $\mu_S(x[x_0, 0, x_2])$

	0	1	2	3	4	5	6	7	8	9	10
$5q+5d$	0	−1	0	0	−1	2	−1	0	0	−1	2
$5q+5d+1$	1	1	−1	0	0	−1	2	−1	0	0	−1
$5q+5d+2$	−1	1	1	−1	0	0	−1	2	−1	0	0
	0	−1	1	1	−1	0	0	−1	2	−1	0
	0	0	−1	1	1	−1	0	0	−1	2	−1
$6q+4d-1$	−1	0	0	−1	1	1	−1	0	0	−1	2
$6q+4d$	2	−1	0	0	−1	1	1	−1	0	0	−1
$6q+4d+1$	−1	2	−1	0	0	−1	1	1	−1	0	0
	0	−1	2	−1	0	0	−1	1	1	−1	0
	0	0	−1	2	−1	0	0	−1	1	1	−1
$6q+5d-1$	−1	0	0	−1	2	−1	0	0	−1	1	1
$6q+5d$	2	−1	0	0	−1	2	−1	0	0	−1	1
$6q+5d+1$	−1	2	−1	0	0	−1	2	−1	0	0	−1
	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
$6q+6d-1$	0	0	0	−1	2	−1	0	0	−1	2	−1

$5q+5d = 6q+3d-1$, $5q+5d+1 = 6q+3d$, $5q+5d+2 = 6q+3d+1$

For $S = \langle 2q, 2q+d, 2q+2d \rangle$

Theorem (C., Ramírez Alfonsín)

For every $i \in \{-1, 0, 1\}$, we consider

$$A_i = \{m(q+d) + i \mid m \in \mathbb{N}\},$$

$$B_i = \{m(q+d) - nd + i \mid m \in \mathbb{N}, n \in \{1, 2, \ldots, \lfloor m/2 \rfloor\}\},$$

$$C_i = A_i \bigcup B_i.$$

Let $\mathfrak{m}_M : \mathbb{N} \longrightarrow \mathbb{N}$ be the *multiplicity function* of a multiset M of $\mathbb{N}$, that is, the function which assigns to each element $x \in \mathbb{N}$ its multiplicity in M.

Let $x_0, i, x_2 \in \mathbb{N}$ such that $i \in \{0, 1\}$ and $i + 2x_2 < a$. Then,

$$\mu_S(x[x_0, i, 0]) = (-1)^i \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1}\right)(x_0)$$

$$\mu_S(x[x_0, i, x_2]) = (-1)^i \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1}\right)(x_0 - x_2), \text{ for } x_2 \geq 1.$$