Wilf's conjecture for numerical semigroups

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Introduction

In 1978 Herbert S. Wilf conjectured an upper bound for the Frobenius number of a numerical semigroup in terms of other invariants of the semigroup.
No counterexample is known and several particular cases of the conjecture have been proved, but it remains unsolved in its full generality.

Setup

Let $g_1, g_2, \dots, g_\nu \in \mathbb{N}$, with $\gcd(g_1, \dots, g_\nu) = 1$.
The **numerical semigroup** generated by these integers is

$$S = \langle g_1, \dots, g_\nu \rangle = \Big\{ \sum_{i=1}^{\nu} \lambda_i g_i,\ \lambda_i \in \mathbb{N} \Big\}.$$

Throughout the talk we assume that $\{g_1, g_2, \dots, g_\nu\}$ is the unique minimal system of generators.

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- The **genus** $g(S) = |\mathbb{N} \setminus S|$
- The integer $n(S) = c(S) - g(S) = |S \cap [0, f(S)]|$

Statement of the conjecture

Conjecture (H.S. Wilf, 1978)

Let S be a numerical semigroup, then

$$c(S) \leq \nu(S) n(S).$$

Known results

The inequality holds in the following cases:

- $\nu(S) \leq 3$ (Fröberg, Gottlieb, Häggkvist, 1987)

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- $f(S) \leq 20$ (Dobbs, Matthews, 2003)
- $n(S) \leq 4$ (Dobbs, Matthews, 2003)
- $n(S) \geq \frac{c(S)}{4}$ (Dobbs, Matthews, 2003)

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The conjecture has been checked by Bras-Amorós by brute force for numerical semigroups with genus $g(S) \leq 50$ (more than $2 \cdot 10^{11}$ semigroups!)

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However, many proofs of the cases mentioned above cannot be generalized to arbitrary semigroups.

An auxiliary result

- An integer $x \in \mathbb{Z}$ is called **pseudo-Frobenius number** of S if $x \notin S$ and $x + s \in S$ for every $s \in S \setminus \{0\}$.

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- The **type** $t(S)$ is the cardinality of the set of pseudo-Frobenius numbers.

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Theorem (Fröberg, Gottlieb, Häggkvist, 1987)

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Corollary

If $t(S) < \nu(S)$ then S satisfies Wilf's conjecture.

Example

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Thus $t(S) = \nu(S)$ and we cannot apply the Corollary.
Nevertheless, S satisfies Wilf's conjecture.

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Let m be the multiplicity of the semigroup. Given $k \in \mathbb{N}$, we define the *k*-**th interval** as

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and we set

$$n_k = \left|\left\{s \in S \cap I_k, s < f(S)\right\}\right|.$$

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Proposition

Let L and n_k be the integers defined above. Then we have:

- $n_0 = 1$;
- $n_k \leq n_{k+1}$ *for* $k \leq L-1$;
- $n_k = 0$ *for* $k \geq L+1$;
- $n(S) = \sum_{k=0}^{L} n_k$.

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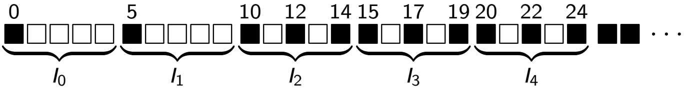

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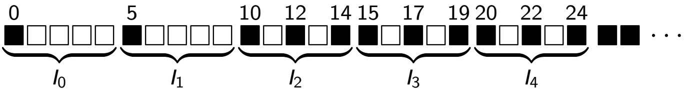

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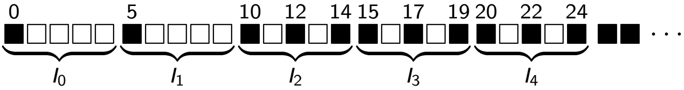

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$n_0 = 1 \qquad n_1 = 1 \qquad n_2 = 3 \qquad n_3 = 3 \qquad n_4 = 2$

Equivalent statement

Define $\rho \in \mathbb{N}$ to be the integer such that

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Proposition

A semigroup S satisfies Wilf's conjecture if and only if

$$\sum_{k=0}^{L-1} (n_k \nu - m) + (n_L \nu - \rho) \geq 0.$$

Semigroups with large embedding dimension

Theorem

Let S be a numerical semigroup such that $2\nu(S) \geq m(S)$. Then S satisfies Wilf's conjecture.

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Idea of the proof. Consider the equivalent form

$$\sum_{k=0}^{L-1}(n_k\nu - m) + (n_L\nu - \rho) \geq 0.$$

For the first term we need to estimate the growth of the sequence $\{n_k\}$; this can be done by means of the Apéry set.
The term $(n_L\nu - \rho)$ needs to be addressed separately.

Semigroups with small multiplicity

Corollary

If $m(S) \leq 8$ then S satisfies Wilf's conjecture.

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Proof. If $m(S) \leq 8$ then we have $\nu(S) \leq 3$ or $2\nu(S) \geq m(S)$.

Semigroups generated by a generalized arithmetic sequence

A **generalized arithmetic sequence** is a sequence of integers of the form

$$m, hm + d, hm + 2d, \ldots, hm + ld$$

with $m, h, d, l \in \mathbb{N}$.

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- $\gcd(m, d) = 1$

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These semigroups have been studied by Matthews (2005).

Semigroups generated by a generalized arithmetic sequence

Proposition

If S is a semigroup generated by a generalized arithmetic sequence, then S satisfies Wilf's conjecture.

An asymptotic result

Wilf's inequality can also be expressed as

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Theorem (Zhai, 2011)

Fix a positive integer k. For every $\varepsilon > 0$ the inequality

$$\frac{n(S)}{c(S)} \geq \frac{1}{k} - \varepsilon$$

is satisfied by all but finitely many semigroups S with embedding dimension $\nu(S) = k$.

A more traditional result

Theorem (Kaplan, 2011)

A numerical semigroup S satisfies Wilf's conjecture if one of the following conditions holds:

- $f(S) < 2m(S)$;
- $2g(S) < 3m(S)$.

Another open problem

In the original paper, Wilf also asked whether the equality

$$c(S) = n(S)\nu(S)$$

holds if and only if the semigroup S is of the form

$$S = \{0, m, m+1, \ldots, \} = \langle m, \ldots, 2m-1 \rangle$$

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In 1987 Fröberg et al. noted that there are other cases in which we have the equality in Wilf's conjecture: for example two-generated semigroups and semigroups of the form

$$S = \langle m, hm+1, hm+2, \dots, hm+(m-1) \rangle$$

Another open problem

Question

Is it true that the equality

$$c(S) = n(S)\nu(S)$$

holds if and only if the semigroup S *satisfies one of the following conditions:*

- $S = \langle a, b \rangle$ *with* a, b *coprime positive integers;*
- $S = \langle m, hm+1, hm+2, \ldots, hm+(m-1) \rangle$ *with* m, h *positive integers.*

Thanks for the attention!