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On some Weierstrass semigroups and the order bound.

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Introduction.

The classical theory of Weierstrass points on projective smooth curves and their associated Weierstrass semigroups is closely related to algebraic-geometric (AG) codes.

Q closed point on X smooth projective curve

↙ ↘

Weierstrass semigroup S $\longleftrightarrow$ Family of AG-codes $\{C_i\}_{i\in\mathbb{N}}$

↑

Feng-Rao function $\nu : S \longrightarrow \mathbb{N}$:
for $i >> 0$ it is a good bound for the minimum distance of C_i

Hence: new interest in studying Weierstrass semigroups.

1. Weierstrass semigroups and deformations.

Let k be an algebraically closed field, let X be a smooth projective curve of genus g defined over k with function field $k(X)$. Let $Q \in X$ be e closed point. For each $n \in \mathbb{N}$:

$$\mathcal{L}(nQ) = \{f \in k(X) \setminus 0 \mid div(f) + nQ \geq 0\} \cup \{0\}$$

is a k vector space of finite dimension $\lambda(nQ)$. By Riemann-Roch the set

$$H(Q) = \{n \in \mathbb{N}^{+} \mid \lambda((n-1)Q) = \lambda(nQ)\}$$

is a proper subset of $\{1, 2, \ldots, 2g\}$ of cardinality g. Its complement

$$S(Q) := \mathbb{N} \setminus H(Q)$$

is a numerical semigroup of genus g (=δ invariant). $S(Q)$ is the same for all but finitely many points Q (in characteristic 0 the generic semigroup is ordinary: $S = \{0, g, g+1 \rightarrow\}$ for almost all points Q). The exceptions are respectively called

Weierstrass points and Weierstrass semigroups.

(Some authors call Weierstrass any semigroup $S(Q)$, $Q \in X$)

It is known that there are non-Weierstrass semigroups.

Example Buchweitz (1980)
$S =< 13, 14, 15, 16, 17, 18, 20, 22, 23 >, \quad g = 16.$

Further examples and results with necessary conditions by Kim, Komeda, Torres and others.

Question How to recognize Weierstrass semigroups?

From now on we shall assume $char\ k = 0$.

One possible way is due to Pinkham (1974).

Theorem [Pinkham] Let S be a numerical semigroup, let $X = Spec(k[S])$ be the associated monomial curve. Then:

$$S \text{ is Weierstrass} \iff X \text{ is smoothable.}$$

This means that there exists a deformation $\pi : Y \longrightarrow \Sigma$ of X

$$\begin{array}{ccc} \pi^{-1}(0) \simeq X & \hookrightarrow & Y \\ \downarrow & & \downarrow \pi \quad (flat) \\ \{0\} & \hookrightarrow & \Sigma \ (base\ space) \end{array}$$

with Σ integral scheme of finite type, such that π admits non-singular fibers.

As a consequence of Pinkham's theorem several semigroups result to be Weierstrass. In particular:

(1) S minimally 3-generated [Shaps]

(2) with multiplicity ≤ 5 [Maclachlan, Komeda]

(3) S with genus $g \leq 8$, or $g = 9$ in particular cases [Komeda]

(4) $Spec(k[S])$ complete intersection. $\diamond$

2. Deformations of monomial curves.

Notation $S =< m_0, \ldots, m_n >$ numerical semigroup,

$P = k[x_0, \ldots, x_n]$, with $weight(x_j) = m_j \quad (0 \leq j \leq n)$;

$k[S] = k[t^{m_0}, t^{m_1}, \ldots, t^{m_n}], \quad X = Spec(k[S])$.

$k[S] = P/I_X, \quad I_X = (f_1, \ldots, f_p)$ $(= I$ for short), f_i homogeneous binomial of degree d_i $(1 \leq i \leq p)$.

For any deformation $\pi : Y \longrightarrow \Sigma$ of X, denote by $I_Y = (F_1, ..., F_p)$ the defining ideal of Y.

Theorem [Schlessinger - Pinkham - Whal]

Let $X = Spec(k[S])$, S numerical semigroup. Then X admits a versal deformation $\pi : Y \longrightarrow \Sigma$. Further there exists a k^*-action on Y extending the usual k^*-action on X. $\diamond$

In case $X = \mathrm{Spec}(k[S])$ a versal deformation (or simply any deformation) can be obtained by an algorithm with a finite number of steps, we shall outline.

- The first step is the construction of a first order infinitesimal deformation of X i.e. with parameter space
$\Sigma = Spec\ k[\varepsilon]/(\varepsilon^2)$

- the n-th step is the lifting of the established deformation with parameter space $Spec\ k[\varepsilon]/(\varepsilon^n)$ to a deformation on $Spec\ k[\varepsilon]/(\varepsilon^{n+1})$.

- By the above theorem we know that the process ends.

Now we point out the main tools.

(2.1) **The vector space T_X^1.**

With the above setting, let

$$\begin{array}{rcccl} \phi: & Hom_{\mathcal{O}_X}(\Omega^1_{P/k}\otimes \mathcal{O}_X, \mathcal{O}_X) & \longrightarrow & Hom_{\mathcal{O}_X}(I/I^2,\ \mathcal{O}_X) \\ & \frac{\partial}{\partial x_i} & \mapsto & g:\ \ g(f)=\left(\frac{\partial f}{\partial x_i}\right) \pmod{I} \end{array}$$

It is well-known that the non-trivial infinitesimal deformations are in one-to-one correspondence with the $\mathcal{O}_X$- module

$$T^1 (= T_X^1) = Cokernel(\phi)$$

the correspondence is given by

$$\mathbf{F} = \begin{pmatrix} f_1 + \varepsilon g_1 \\ \dots \\ f_p + \varepsilon g_p \end{pmatrix} \longleftrightarrow \begin{pmatrix} g: I/I^2 \longrightarrow \mathcal{O}_X \\ f_i \mapsto g_i (\mathrm{mod}\ I) \\ (i = 1, \dots, p) \end{pmatrix}.$$

(2.2) T^1_X for monomial curves.

We have:

(a) $T^1 = \bigoplus_{\ell\in\mathbb{Z}} T^1(\ell)$ is a *$\mathbb{Z}$-graded finite dimensional* *k-vector space* .

(b) $g \in T^1(\ell) \Longleftrightarrow v(g(f_i)) = \deg(f_i) + \ell \quad (i = 1, \ldots, p)$, where $v : k(t) \longrightarrow \mathbb{Z}$ is the usual valuation.

(c) Let $\Delta_i := x_i \left(\frac{\partial}{\partial x_i}\right)$; then

$$g \in T^1(\ell) \Longrightarrow \begin{cases} g = t^\ell \sum_{i=1}^n \lambda_i \Delta_i, \quad \lambda_i \in k, \\ \\ g(f_i) = 0 \ \ \forall\ i \ such\ that \ \ d_i + \ell \notin S. \end{cases}$$

(2.3) Construction of a basis for T^1. We have pointed out a method to find a basis for T^1 in the case of monomial curves.

This can be done in an easy way starting from the Jacobian matrix $J_X = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{i=1,\dots,p \\ j=0,\dots,n}}$.

(a) The evaluation $J_X(1)$ of J_X at the point $Q(1,\dots,1) \in X$ is useful to calculate $dim_k\, T^1(\ell)$ for each $\ell \in \mathbb{Z}$ (Buchweitz, Pinkham, Rim).

(b) If $dim_k\, T^1(\ell) > 0$ an element $t^\ell(\sum_{1=1}^n \lambda_i \Delta_i) \in T^1(\ell)$, can be found by imposing the vector $v = (\lambda_1, \dots, \lambda_n) \neq 0$ to be orthogonal to every row of a suitable $(n-1) \times n$ submatrix M_ℓ of $J_X(1)$ of rank $(n-1)$ formed by columns $C_2, \dots, C_{n+1}$ of $J_X(1)$ and by $(n-1)$ independent rows. Hence can choose

$v =$ exterior product of the rows of M_ℓ.

(c) When $(f_1, \dots, f_p)$ have a syzygy matrix ρ_0 with all the entries of $\rho_0(Q) \in \{-1, 0, 1\}$, this method gives a basis $\{\mathbf{g_1}, \dots, \mathbf{g_h}\}$ of T^1 such that $g_j(f_i)(Q) \in \{-1, 0, 1\}\ \forall i, j$.

(2.4) **Condition of flatness.**

(Well-known) Given the diagram

$$\begin{array}{ccc} \pi^{-1}(0) \simeq X & \hookrightarrow & Y \\ \downarrow & & \downarrow \pi \quad (flat) \\ \{0\} & \hookrightarrow & \Sigma \ (base\ space) \end{array}$$

where $I_X = (f_1, \ldots, f_p),\ I_Y = (F_1, \ldots, F_p)$:

the map π is flat $\iff$ every relation $\sum_1^k r_i f_i = 0$ can be lifted to a relation $\sum_1^k R_i F_i = 0$,

($r_i, f_j \in k[x_0, \ldots, x_k],\ R_i, F_j \in k[x_0, \ldots, x_k] \otimes \mathcal{O}_{\Sigma,0}$).

(2.5) **Algorithm (outline).**

Step 1. Denote by $\mathbf{f} := (f_1, \ldots, f_p)^T$.

Given homogeneous elements $g_1, \ldots, g_h \in \bigoplus_{\ell<0} T^1(\ell)$, (a basis of $\bigoplus_{\ell<0} T^1(\ell)$ to obtain a versal deformation) assign a parameter U_j to each g_j, with $weight(U_j) = -deg(g_j)$ and consider

$$g := U_1 g_1 + \cdots + U_h g_h, \qquad \mathbf{g_1} := g(\mathbf{f}) = \big(g(f_1), \ldots, g(f_p)\big)^T$$

Let ρ_0 be a $(m \times p)$ syzygy matrix for $\mathbf{f}$: by construction

$$\rho_0 \mathbf{g_1} \in I_X^m.$$

there exists a matrix ρ_1 $(m \times p$ as $\rho_0)$ with entries $\in k[U_1, \ldots, U_h, x_0, \ldots, x_n])$ such that

$$(\rho_0 + \varepsilon \rho_1)(\mathbf{f} + \varepsilon \mathbf{g_1}) \equiv 0 \ (mod\ \varepsilon^2).$$

Therefore the components $(F_1, \ldots, F_p)$ of $\mathbf{F} = \mathbf{f} + \varepsilon \mathbf{g_1}$ generate the ideal I_{Y_1} of a first order infinitesimal deformation

$$\pi_1 : Y_1 \longrightarrow \Sigma_1 \simeq Spec\, \Big(k[U_1, \ldots, U_h]/(U_1, \ldots, U_h)^2 \Big).$$

Step 2. By repeating the procedure on $Spec\, k[\varepsilon]/(\varepsilon)^3$, we find ρ_2 $(m \times p)$ and $\mathbf{g_2}$ (entries $\in k[x_0, \ldots, x_n, U_1, \ldots, U_h]$), such that

$$(\rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2)(\mathbf{f} + \varepsilon \mathbf{g_1} + \varepsilon^2 \mathbf{g_2}) \equiv 0 \ (mod\ \varepsilon^3).$$

To solve this equation we must impose several conditions $a_{21}, \ldots, a_{2s}$ on the variables $U_1, \ldots, U_h$.
But the above quoted Theorem assures that there exists a solution: it allows to lift π_1 to a deformation $\pi_2 : Y_2 \longrightarrow \Sigma_2$ where

$$\Sigma_2 \simeq Spec\, k[U_1, \ldots, U_h]/\Big((U_1, \ldots, U_h)^3 \cap (a_{21}, \ldots, a_{2s})\Big).$$

We know the algorithm ends in a finite number, say N, of steps. Hence

Step N. Get a deformation $\pi : Y \longrightarrow \Sigma$ defined by

$$\mathbf{F} = \mathbf{f} + U_1 g_1 + \cdots + U_h g_h + U_1^2 h_{11} + \cdots + U_h^N h_{N \ldots N}.$$

Let $\Sigma = Spec(A)$, substitute U_i with $U_i x_{n+1}^{weight(U_i)}$ and let

$$R := A[x_0, \ldots, x_{n+1}]/(F_1, \ldots, F_P).$$

The morphism $\tilde{\pi} : Proj(R) \longrightarrow \Sigma$ induced by π is proper, flat, with fibers reduced projective curves. The generic fiber has only one regular point $Q_\infty(t^{m_0}, \ldots, t^{m_n}, 0)$ at infinity. If a fiber $\mathcal{C}$ is smooth, then the semigroup associated to the pair $(\mathcal{C}, Q_\infty)$ is Weierstrass is equal to S (Pinkham). $\diamond$

(2.6) **Example.**

Let $S =< 4, 9, 11 >$, $X = Spec\ (k[t^4, t^9, t^{11}])$.

We show how to construct a deformation of X. We have $I_X = (f_1, f_2, f_3)$, where $deg(f_1, f_2, f_3) = (20, 22, 27)$,

$$f_1 = x_0^5 - x_1 x_2,\ f_2 = x_0 x_1^2 - x_2^2,\ f_3 = -x_1^3 + x_0^4 x_2$$

$$J_X(1) = \begin{pmatrix} 5 & -1 & -1 \\ 1 & 2 & -2 \\ 4 & -3 & 1 \end{pmatrix}, \rho_0 = \begin{pmatrix} -x_2 & x_1 & x_0 \\ x_1^2 & -x_0^4 & -x_2 \end{pmatrix}.$$

[resp. valuation of the jacobian matrix at $Q(1, 1, 1)$ and syzygy matrix]. Let $\Delta_i := x_i \frac{\partial}{\partial x_i}$, $i = 0, 1, 2$ (degree 0 derivations).

We have $dim_k T^1(\mathcal{O}_X) = 17$ and $T^1(\mathcal{O}_X)$ is $\mathcal{O}_X$-generated by

$$\left[\begin{array}{ll} g_1 = t^{-18}(\Delta_1 - \Delta_2) \in T^1(-18) & v = (1, -1) \perp (-1, -1) \\ g_2 = t^{-16}(\Delta_1 + \Delta_2) \in T^1(-16) & v = (1, 1) \perp (2, -2) \\ g_3 = t^{-11}(\Delta_1 + \Delta_2) \in T^1(-11) & \end{array} \right.$$

$$g_1(\mathbf{f}) = \begin{pmatrix} 0 \\ x_0 \\ -x_1 \end{pmatrix},\ g_2(\mathbf{f}) = \begin{pmatrix} x_0 \\ 0 \\ x_2 \end{pmatrix},\ g_3(\mathbf{f}) = \begin{pmatrix} x_1 \\ 0 \\ x_0^4 \end{pmatrix}$$

Step 1

Assign to each g_i a parameter U_i and let $\mathbf{g_1} := g(\mathbf{f})$, where
$g = U_1 g_1 + U_2 g_2 + U_3 g_3, \quad weight(U_1, U_2, U_3) = (18, 16, 11)$.
Get a first order infinitesimal deformation of X

$$\pi_1 : Y_1 \longrightarrow \Sigma = Spec\ k[\varepsilon, \varepsilon U_1, \varepsilon U_2, \varepsilon U_3]/(\varepsilon^2)$$

with I_{Y_1} generated by the rows of $\mathbf{F_1} = \mathbf{f} + \varepsilon \mathbf{g_1}$.
Further the matrix $\rho_0 g$ results equal to $-\rho_1 \mathbf{f}$ with

$$\rho_1 = \begin{pmatrix} -U_3 & 0 & 0 \\ U_1 & -U_2 & U_3 \end{pmatrix},$$

hence $\rho_0 + \varepsilon \rho_1$ is a syzygy matrix for $\mathbf{F_1}$.

Step 2. Now look for a lifting $\mathbf{F_2}$ defining a variety Y_2 and a matrix ρ_2 such that

$$\mathbf{F_2} = \mathbf{f} + \varepsilon \mathbf{g_1} + \varepsilon^2 \mathbf{g_2} \quad \text{and} \quad R_2 = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2$$

satisfy $F_2 R_2 = 0 \ (mod\ (\varepsilon^3))$, i.e., $\rho_0 \mathbf{g_2} + \rho_1 \mathbf{g_1} + \rho_2 \mathbf{f} \equiv 0$.

Since $\rho_1 \mathbf{g_1} = \begin{pmatrix} -x_2 & x_1 & x_0 \\ x_1^2 & -x_0^4 & -x_2 \end{pmatrix} \begin{pmatrix} 0 \\ -U_3^2 \\ -U_2U_3 \end{pmatrix} = -\rho_0 \mathbf{g_2}$,

with $\mathbf{g_2} = \begin{pmatrix} 0 \\ U_3^2 \\ U_2U_3 \end{pmatrix}$, we can choose $\rho_2 = 0$. Hence

$\pi_2 : Y_2 \longrightarrow Spec\ \left(k[\varepsilon, \varepsilon U_1, \varepsilon U_2, \varepsilon U_3]/(\varepsilon^3)\right)$ is flat.

Since $\rho_1 \mathbf{g_2} = 0$, the algorithm ends at Step 2.

Substitute εU_i with $x_4^{weight(U_i)} U_i$ for each $i = 1, 2, 3$ and let $F_i,\ (i = 1, 2, 3)$ be the weighted homogeneous rows of

$$\mathbf{F} = \mathbf{f} + \begin{pmatrix} x_0 x_4^{16}\ U_2 & + & x_1 x_4^{11}\ U_3 \\ x_0 x_4^{18}\ U_1 & + & x_4^{22}\ U_3^2 \\ x_1 x_4^{18}\ U_1 + x_2 x_4^{16}\ U_2 & + & x_0^4 x_4^{11}\ U_3 + x_4^{27}\ U_2 U_3 \end{pmatrix}.$$

With $B = k[x_0, \ldots, x_4, U_1, U_2, U_3]/(F_1, F_2, F_3)$ we get

$$\tilde{\pi} : Proj(A) \longrightarrow Spec\ \big(k[U_1, U_2, U_3]\big)$$

whose fibres are weighted projective curves with one smooth point $Q_\infty(t^4, t^9, t^{11}, 0)$ at infinity.

One can easily check that the generic fibre is non-singular. $\diamond$

3. AS semigroups.

We apply the above tools to numerical semigroups S minimally generated by an arithmetic sequence (AS semigroups):

$S =< m_0, ..., m_n >$ with $m_i = m_0 + id, \quad d \geq 1, i = 1, \ldots, n.$

Let $a, b \in \mathbb{N}$ be such that

$$m_0 = an + b \text{ , with } a \geq 1, \; 1 \leq b \leq n$$

and let $\mu = a + d$.
The ideal I defining the curve $X = Spec\ k[S]$ is generated by the 2×2 minors of the following two matrices:

$$A = \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \ldots & x_{n-1} & x_n \end{pmatrix}, \quad A' := \begin{pmatrix} x_n^a & x_0 & \ldots & x_{n-b} \\ x_0^\mu & x_b & \ldots & x_n \end{pmatrix}$$

A minimal set of generators for I is formed by the $\binom{n}{2}$ maximal minors of the matrix A (which define the affine cone on the rational normal curve of $\mathbb{P}^n$) and the $(n - b + 1)$ maximal minors $M_{1,j}$ of the matrix A'.

As regards the module $T^1 = T^1(\mathcal{O}_X)$ we have:

Lemma

(1) If $\ell < -\mu m_0$, then $T^1(\ell) = 0$ except the case

$\ell = -(\mu+1)m_0$, when $b = n$.

(2) $dim_k\, T^1(-\mu m_0) = 1$.

Theorem

If $b = 1$ or $b = n$, then the semigroup is Weierstrass.

Proof (outline). If $b = 1$ the ideal I is determinantal generated by the (2×2) minors of A'. In this case one can deduce the smoothability of the curve X by a result of M.Shaps on determinantal ideals. Nevertheless by the above algorithm we easily construct a deformation Y with smooth fibres using a basis of $T^1(-\mu m_0)$. The ideal I_Y is again determinantal, generated by the (2×2) minors of the deformed matrix

$$A'_{def} = \begin{pmatrix} x_n^a & x_0 & \dots & x_{n-1} \\ x_0^{\mu} - U x_{n+1}^{\mu m_0} & x_1 & \dots & x_n \end{pmatrix}.$$

If $b = n$ we proved the curve X is smoothable by constructing a 1-parameter family of curves with smooth fibres

$$\pi : Y \longrightarrow Spec\ k[U].$$

This deformation is related to $g \in T^1(-(\mu+1)m_0)$: the defining ideal of Y is

$$I_Y = \left(f_1, \ldots, f_{\binom{n}{2}},\ x_n^{a+1} - x_0^{\mu+1} + U\right)$$

(where $f_1, \ldots, f_{\binom{n}{2}}$ are as above the maximal minors of $A = \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \ldots & x_{n-1} & x_n \end{pmatrix}$, hence define the cone over the rational normal curve $C \subseteq \mathbb{P}^n$).

Corollary

Let $S = < m_0, \ldots, m_n >$ be an AS-semigroup of embedding dimension ≤ 5. Then S is Weierstrass. In fact:

(1) If $n = 2$, S is Weierstrass since every curve $X \subseteq \mathbb{A}^3$ is smoothable.

(2) If $n = 3$, S is Weierstrass; in fact $b \in \{1, 2, 3\}$ and the remaining case $b = 2$ is known since X is Gorenstein of codimension 3 (Buchsbaum-Eisenbud 1977).

(3) If $n = 4$, $(X \subseteq \mathbb{A}^5)$ for the cases $b \in \{2, 3\}$ we have found suitable deformations with generic smooth fibre.

Conjecture

The result is true for any embedding dimension (work in progress).

4. Order bound for AS-semigroups.

Let X be a projective smooth algebraic curve defined over a field k, of genus g and let $k(X)$ be its function field. Given a k-rational Weierstrass point $Q \in X$, let $S = S(Q)$ be the corresponding Weierstrass semigroup: a family of *codes* can be associated to (X, Q) as follows.
Let $P_1, ..., P_m$ be distinct k-rational points of X, $P_j \neq Q$ for each j: for each $n \in \mathbb{N}$, consider the finite dimensional k-vector space

$$\mathcal{L}(nQ) = \{f \in k(X) \setminus 0 \mid div(f) + nQ \geq 0\} \cup \{0\}$$

and define

$$\Phi : \mathcal{L}(nQ) \longrightarrow k^m, \qquad \Phi(f) = (f(P_1), ..., f(P_m)).$$

Then $(Im\ \Phi)^{\perp} := C_n$ is the one-point AG code of order m associated to Q and to the divisor $P_1 + ... + P_m$.

A good measure for the *minimum distance* $d(C_n)$ of an AG code C_n, is the Feng-Rao order bound, denoted $d_{ORD}(C_n)$ which depends only on the semigroup S: for $s_j \in S$, let

$$\nu(s_j) \;:=\; \#\{(s_h, s_k) \in S^2 \mid s_j = s_h + s_k\}$$

the Feng-Rao order bound of the code C_n is defined as

$$d_{ORD}(C_n) \;:=\; min\{\nu(s_j) \mid j > n\} \leq d(C_n).$$

For S ordinary the sequence $\{\nu(s_j)\}_{j\in\mathbb{N}}$ is non-decreasing and

$$d_{ORD}(i) = \nu(s_{i+1}) \;\; for \quad i \geq 0.$$

In general there exists $m \in \mathbb{N}$ such that

$$\nu(s_m) > \nu(s_{m+1}) \;\; and \;\; \nu(s_{m+k}) \leq \nu(s_{m+k+1}) \;\; \forall\, k \geq 1.$$

Then: $d_{ORD}(C_n) = \nu(s_{n+1})$ for each code C_n with $n \geq m$.

We have calculated $d_{ORD}(C_n)$ for AG-codes related to AS-semigroups. Let:

$$c = min\ \{r \in S \mid r + \mathbb{N} \subseteq S\},\ \textit{the conductor}$$
$$\delta = \max\{s_i \in S \mid s_i < c\},\ \textit{the dominant}$$
$$g\ \ \textit{the genus.}$$

Theorem

$$d_{ORD}(\mathcal{C}_i) = \left[\begin{array}{llrll} s_i + 2 - 2g & if & s_m \leq & s_i & \leq 2\delta - 1 \\ 2(s_i + 1 - \delta - g) & if & 2\delta \leq & s_i & \leq \delta + c - 1 \\ 2c - 2g & if & \delta + c \leq & s_i & \leq 2c - 2 \\ 2c - 2g + k & if & k > 0, & s_i & = 2c - 2 + k \end{array} \right.$$

where

$$s_m = \left[\begin{array}{lll} 2c - d - 2 & if & 2 \leq d < m_0 \ \ and \ \ b \neq 2 \\ 2c - m_0 - 2 & otherwise & \end{array} \right. \qquad \diamond$$

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