Some questions on numerical semigroups with type two

Aureliano M. Robles-Pérez

Universidad de Granada

A talk based on joint works with José Carlos Rosales

Iberian meeting on numerical semigroups - Vila Real 2012
July 18-20, 2012

- Let S be a numerical semigroup.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition

- $x \in \mathbb{Z} \setminus S$ *is a* pseudo-Frobenius number *of* S *if* $x + s \in S$ *for all* $s \in S \setminus \{0\}$.
- $\mathrm{PF}(S)$ *is the set of pseudo-Frobenius numbers of* S.
- $\mathrm{t}(S) = \sharp\mathrm{PF}(S)$ *(*type *of* S*).*

- $\mathrm{F}(S) = \max(\mathbb{Z} \setminus S) = \max(\mathrm{PF}(S))$ (*Frobenius number* of S).

Motivation: symmetric numerical semigroups

Definition

- *S is a* symmetric *numerical semigroup if S has type equal to one.*

Corollary

- *S is a* symmetric *numerical semigroup if and only if $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$.*

- $\mathrm{g}(S) = \sharp(\mathbb{N} \setminus S)$ (*genus* of S).

Lemma

- *S is a* symmetric *numerical semigroup if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.*

Proposition

- *If two symmetric numerical semigroups have the same Frobenius number, then they have the same genus.*
- $\{F\}$ *is the set of pseudo-Frobenius numbers of a symmetric numerical semigroup if and only if* F *is an odd integer greater than or equal to* -1*.*

- $S = \mathbb{N}$ has Frobenius number equal to -1.
- $\{0, \frac{F+1}{2}, \rightarrow\} \setminus \{F\}$ is a symmetric numerical semigroup with Frobenius number F (if F is an odd integer greater than or equal to 1).

  ($\rightarrow$ means that every integer greater than $\frac{F+1}{2}$ belongs to the set.)

Questions

1. What conditions must satisfy two integers g_1, g_2 in order to there exists a numerical semigroup S such that $\mathrm{PF}(S) = \{g_1, g_2\}$?
2. If two numerical semigroups of type two have the same Frobenius number, what are their genera?
3. If two numerical semigroups of type two have the same set of pseudo-Frobenius numbers, what are their genera?

Question 1: a particular case with a quick answer (pseudo-symmetric numerical semigroups)

Definition

- *S is a* pseudo-symmetric *numerical semigroup if* $\mathrm{PF}(S) = \left\{\frac{\mathrm{F}(S)}{2}, \mathrm{F}(S)\right\}$.

Lemma

- *S is a* pseudo-symmetric *numerical semigroup if and only if* $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$.

Proposition

- *If g_2 is an even positive integer, then there always exists a numerical semigroup S such that* $\mathrm{PF}(S) = \left\{\frac{g_2}{2}, g_2\right\}$.

- $\{0, \frac{F}{2}+1, \rightarrow\} \setminus \{F\}$ is a pseudo-symmetric numerical semigroup with Frobenius number F (if F is an even positive integer).

- Let g_1, g_2 be two positive integers such that $g_1 < g_2$ and $g_1 \neq \frac{g_2}{2}$.
- We look for a numerical semigroup S such that $\mathrm{PF}(S) = \{g_1, g_2\}$.

Idea

- Determine strictly necessary elements of S.
- Remove as many elements as we can.

- Let g_1, g_2 be two positive integers such that $g_1 < g_2$ and $g_1 \neq \frac{g_2}{2}$.
- Let S be a numerical semigroup.

Lemma

- *If $x \in \mathbb{N} \setminus S$, then there exists $g \in \mathrm{PF}(S)$ such that $g - x \in S$.*

- Let S be a numerical semigroup such that $\mathrm{PF}(S) = \{g_1, g_2\}$.

Lemma

- $g_1 = \max\left\{x \in \mathbb{N} \setminus S \mid x \neq \frac{g_2}{2} \text{ and } g_2 - x \notin S\right\}$.
- $\frac{g_2}{2} < g_1$.
- $g_2 - g_1 \notin S$.
- $2g_1 - g_2 \in S$.
- *If g_2 is even, then $g_1 - \frac{g_2}{2} \in S$.*

- Let g_1, g_2 be two positive integers such that $g_1 < g_2$ and g_2 is odd.

Proposition

- *There exists a numerical semigroup S such that $\mathrm{PF}(S) = \{g_1, g_2\}$ if and only if $\frac{g_2}{2} < g_1$, $(2g_1 - g_2) \nmid g_1$, $(2g_1 - g_2) \nmid g_2$, and $(2g_1 - g_2) \nmid (g_2 - g_1)$.*

Proof.

- $m = 2g_1 - g_2 = \mathrm{m}(S) = \min(S \setminus \{0\})$.
- $\mathrm{R}(g_i) = \{g_i - km \mid k \in \mathbb{N}\}$, $i \in \{1, 2\}$.
- $S = \left(\langle m \rangle \cup \left\{\frac{g_2+1}{2}, \rightarrow\right\}\right) \setminus (\mathrm{R}(g_1) \cup \mathrm{R}(g_2))$.

□

- Let g_1, g_2 be two positive integers such that $g_1 < g_2$ and g_2 is even.

Proposition

- *There exists a numerical semigroup S such that $\mathrm{PF}(S) = \{g_1, g_2\}$ if and only if $\frac{g_2}{2} < g_1$, $(g_1 - \frac{g_2}{2}) \nmid g_1$, $(g_1 - \frac{g_2}{2}) \nmid g_2$, and $(g_1 - \frac{g_2}{2}) \nmid (g_2 - g_1)$.*

Proof.

- $m = g_1 - \frac{g_2}{2} = \mathrm{m}(S) = \min(S \setminus \{0\})$.
- $\mathrm{R}(g_i) = \{g_i - km \mid k \in \mathbb{N}\}$, $i \in \{1, 2\}$.
- $S = \left(\langle m \rangle \cup \left\{\frac{g_2}{2} + 1, \rightarrow\right\}\right) \setminus (\mathrm{R}(g_1) \cup \mathrm{R}(g_2))$.

□

Example: a numerical semigroup S with $\mathrm{PF}(S) = \{7, 9\}$

- $m = 2 \times 7 - 9 = 5$.
- $\mathrm{R}(7) = \{7, 2\}$, $\mathrm{R}(9) = \{9, 4\}$.
- $S = \left(\langle 5 \rangle \cup \left\{\frac{9+1}{2}, \rightarrow\right\}\right) \setminus \{2, 4, 7, 9\} = \{0, 5, 6, 8, 10, \rightarrow\} = \langle 5, 6, 8 \rangle$.

Example: a numerical semigroup S with $\mathrm{PF}(S) = \{9, 10\}$

- $m = 9 - \frac{10}{2} = 4$.
- $\mathrm{R}(9) = \{9, 5, 1\}$, $\mathrm{R}(10) = \{10, 6, 2\}$.
- $S = \left(\langle 4 \rangle \cup \left\{\frac{4}{2} + 1, \rightarrow\right\}\right) \setminus \{1, 2, 5, 6, 9, 10\} = \{0, 4, 7, 8, 11, \rightarrow\} = \langle 4, 7, 13 \rangle$.

Question 2: preliminaries (bounding the genus by the Frobenius number)

- Let S be a numerical semigroup.
- $\mathrm{n}(S) = \sharp\{s \in S \mid s < \mathrm{F}(S)\}$.

Lemma

- $\mathrm{g}(S) + \mathrm{n}(S) = \mathrm{F}(S) + 1$.
- $\mathrm{g}(S) \leq \mathrm{t}(S)\mathrm{n}(S)$.
- $\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}$.
- $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$ *if and only if* $\mathrm{t}(S) = 1$ *(symmetric numerical semigroups).*

Proposition

- *If* $\mathrm{t}(S) = 2$, *then* $\frac{\mathrm{F}(S)+2}{2} \leq \mathrm{g}(S) \leq \frac{2(\mathrm{F}(S)+1)}{3}$.

- Let $\langle n_1, n_2, \ldots, n_e \rangle$ be the minimal system of generators of S.
- $\mathrm{e}(S) = \sharp\{n_1, n_2, \ldots, n_e\}$ (*embedding dimension* of S).

Lemma

- *If* $\mathrm{e}(S) = 3$ *and* n_1, n_2, n_3 *are pairwise relatively prime minimal generators of* S*, then* $\mathrm{t}(S) = 2$*.*

- Let $k \in \mathbb{N} \setminus \{0\}$.

Proposition

- $S = \langle 3, 3k+1, 3k+2 \rangle$ *is a numerical semigroup with type two and* $\mathrm{g}(S) = \frac{2(\mathrm{F}(S)+1)}{3}$*.*
- *Moreover,* $\mathrm{PF}(S) = \{\mathrm{F}(S), \mathrm{F}(S) - 1\} = \{3k-1, 3k-2\}$*.*

- From the quick answer to question 1.
- Let F be an even positive integer.

Proposition

- *$S = \left\{0, \frac{F}{2}+1, \rightarrow\right\} \setminus \{F\}$ is a numerical semigroup with type two and $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$.*
- *Moreover, $\mathrm{PF}(S) = \left\{F, \frac{F}{2}\right\}$.*

Question 2: a first answer

- Let $k \in \mathbb{N} \setminus \{0\}$.
- Let $F = 6k - 4$.

Proposition

- $S = \langle 3, 6k-2, 6k-1 \rangle$ *is a numerical semigroup with type two and* $\mathrm{g}(S) = 4k-2$.
- *Moreover,* $\mathrm{PF}(S) = \{6k-5, 6k-4\}$.

Proposition

- $S = \{0, 3k-1, \rightarrow\} \setminus \{6k-4\}$ *is a numerical semigroup with type two and* $\mathrm{g}(S) = 3k-1$.
- *Moreover,* $\mathrm{PF}(S) = \{3k-2, 6k-4\}$.

Question 3: preliminaries

- Let F be an odd integer greater than or equal to 1.

Lemma (symmetric numerical semigroups)

- *$S = \{0, \frac{F+1}{2}, \rightarrow\} \setminus \{F\}$ is a numerical semigroup with $\mathrm{t}(S) = 1$ and Frobenius number F.*

- Let S be a numerical semigroup with $\mathrm{t}(S) = 1$.

Lemma

- *If x is a minimal generator of S such that $x < \mathrm{F}(S)$, then $\mathrm{PF}(S \setminus \{x\}) = \{\mathrm{F}(S), x\}$ if and only if $2x - \mathrm{F}(S) \in S$.*

- Let F be an odd integer greater than or equal to -1.

Types different to 2

- if $\mathrm{F}(S) = -1$, then $S = \mathbb{N}$ with $\mathrm{t}(\mathbb{N}) = 1$.
- if $\mathrm{F}(S) = 1$, then $S = \langle 2, 3 \rangle$ with $\mathrm{t}(\langle 2, 3 \rangle) = 1$.
- if $\mathrm{F}(S) = 3$, then $S = \langle 2, 5 \rangle$ with $\mathrm{t}(\langle 2, 5 \rangle) = 1$ or $S = \langle 4, 5, 6, 7 \rangle$ with $\mathrm{t}(\langle 4, 5, 6, 7 \rangle) = 3$.

- Let F be an odd integer greater than or equal to 5.

Proposition

- $S = \left\{0, \frac{F+1}{2}, \rightarrow\right\} \setminus \{F, F-1\}$ *is a numerical semigroup of type two with* $\mathrm{F}(S) = F$, $\mathrm{g}(S) = \frac{\mathrm{F}(S)+3}{2} = \left\lceil \frac{\mathrm{F}(S)+2}{2} \right\rceil$, *and* $\mathrm{PF}(S) = \{F, F-1\}$.

Question 3: answer

- Let $k \in \mathbb{N} \setminus \{0\}$.
- Let $F = 6k-1$.

Remark

- *If S is a numerical semigroup of type two with Frobenius number F, then $3k+1 \leq \mathrm{g}(S) \leq 4k$.*

Corollary

- $S = \langle 3, 6k+1, 6k+2 \rangle$ *is a numerical semigroup with* $\mathrm{PF}(S) = \{6k-1, 6k-2\}$ *and* $\mathrm{g}(S) = 4k$*.*
- $S' = \{0, 3k, \rightarrow\} \setminus \{6k-1, 6k-2\}$ *is a numerical semigroup with* $\mathrm{PF}(S') = \{6k-1, 6k-2\}$ *and* $\mathrm{g}(S') = 3k+1$*.*

Question 2: a second (and better) answer

- Let $k \in \mathbb{N} \setminus \{0\}$.
- Let $n \in \{3k+1, \ldots, 4k\}$.
- Let $r = 4k - n$.

Proposition

- *$S = \langle 3, 6k+1-3r, 6k+2 \rangle$ is a numerical semigroup of type two with $\mathrm{F}(S) = 6k-1$ and $\mathrm{g}(S) = n$*
- *Moreover, $\mathrm{PF}(S) = \{6k+1-3(r-1), 6k-1\}$.*

References

V. Barucci, D.E. Dobbs and M. Fontana.
"Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains"
Mem. Amer. Math. Soc. 598 (1997).

A.M. Robles-Pérez and J.C. Rosales.
Modular translations of numerical semigroups.
Semigroup Forum. DOI 10.1007/s00233-012-9372-8.

A.M. Robles-Pérez and J.C. Rosales.
Modular retractions of numerical semigroups.
Submitted.

J.C. Rosales and P.A. García-Sánchez.
"Numerical semigroups", Developments in Mathematics, vol. **20**.
Springer, New York, 2009.

THANK YOU VERY MUCH FOR YOUR ATTENTION!