Identities for the degrees of syzygies in numerical semigroup rings

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Introduction

We will prove some theorems from Fel, New identities for degree of syzygies in numerical semigroups.

Some commutative algebra

Example Let $S = \langle 4, 7, 9 \rangle$, $R = k[S] = k[t^4, t^7, t^9]$. $k[S]$ is a graded ring ($\deg t = 1$) and has a Hilbert series, $H_R(t) = \sum_{i \geq 0} \dim_k R_i t^i$. Now $\dim_k R_i = 1$ if $i \in S$, $\dim_k R_i = 0$ if $i \notin S$. Thus $H_R(t) = 1 + t^4 + t^7 + t^8 + t^9 + t^{11}/(1-t)$. If $S = \langle n_1, \ldots, n_k \rangle$, the Hilbert series of $k[S]$ is $p(t) + t^c/(1-t)$, where $p(t)$ is a polynomial.

Let $A = k[X, Y, Z]$, $\deg X = 4, \deg Y = 7, \deg Z = 9$, and let $\Phi_0 \colon A \longrightarrow R$ be defined by $X \mapsto t^4, Y \mapsto t^7, Z \mapsto t^9$.

We have $A \xrightarrow{\Phi_0} R \longrightarrow 0$ with kernel $(x^4 - yz, z^2 - xy^2, y^3 - x^3z)$ (the first syzygies) corresponding to $4 \cdot 4 = 9 + 7$, $2 \cdot 9 = 4 + 2 \cdot 7$, $3 \cdot 7 = 3 \cdot 4 + 9$. The first syzygies have degree $16, 18, 21$, respectively.

Now we have $A[-16] \oplus A[-18] \oplus A[-21] \xrightarrow{\Phi_1} A \longrightarrow R \longrightarrow 0$, where $A[-n]_i = A_{n+i}$, and Φ_1 has degree 0. The first Betti numbers are $\beta_{1,16} = \beta_{1,18} = \beta_{1,21} = 1$.

The relations between the first syzygies (the second syzygies) have degree 25 and 30, corresponding to 9+16=7+18=21+4 and 14+16=12+18=21+9.

Then we get $0 \longrightarrow A[-25] \oplus A[-30] \xrightarrow{\Phi_2} A[-16] \oplus A[-18] \oplus A[-21] \xrightarrow{\phi_1} A \longrightarrow R \longrightarrow 0$, and Φ_2 has degree 0. The second Betti numbers are $\beta_{2,25} = \beta_{2,30} = 1$.

Now we can calculate the Hilbert series in another way. Let $\dim_k A_n = a_n$ and $\dim_k R_n = r_n$. If we concentrate on degree n in the exact sequence, we have an exact sequence of k-spaces, so the alternating sum of dimensions is 0. Thus $r_n = a_n - (a_{-16+n} + a_{-18+n} + a_{-21+n}) + a_{-25+n} + a_{-30+n}$. If we multiply with t^n and sum, we get $\sum r_n t^t = (1 - t^{16} - t^{18} - t^{21} + t^{25} + t^{30}) \sum a_n t^n$.

If $\deg X = n$, then the Hilbert series of $k[X]$ is $1 + t^n + t^{2n} + \cdots = 1/(1 - t^n)$. Now $A = k[X] \otimes k[Y] \otimes k[Z]$, so A has Hilbert series $1/((1 - t^4)(1 - t^7)(1 - t^9))$. Thus R has Hilbert series $(1 - t^{16} - t^{18} - t^{21} + t^{25} + t^{30})/((1 - t^4)(1 - t^7)(1 - t^9))$.

Theorem

If $S = \langle n_1, \ldots, n_k \rangle$, the the Hilbert series of $k[S]$ is $\sum_{i=0}^{k-1} \sum_j (-1)^i \beta_{i,j} t^j / \prod_{i=1}^{k} (1 - t^{n_i})$.

Fel's result

Theorem (Fel)

Let $S = \langle n_1, \ldots, n_k \rangle$. *Then* $\sum_{i=0}^{k-1} \sum_j (-1)^i \beta_{i,j} j^r = 0$ *for* $r = 1, \ldots, k-2$ *and* $\sum_{i=0}^{k-1} \sum_j (-1)^i \beta_{i,j} j^{k-1} = (-1)^{k-1}(k-1)! \prod_{i=1}^{k} n_i$.

Proof

Using the two forms of the Hilbert series

$$p(t) + t^c/(1-t) = ((1-t)p(t) + t^c)/(1-t) =$$

$$\sum_{i=0}^{k-1} \sum_j (-1)^i \beta_{i,j} t^j / \prod_{i=1}^{k} (1-t^i)$$

we get
$\sum_{i=0}^{k-1} \sum_j (-1)^i \beta_{i,j} t^j = (1-t)^{k-1}(p(t)(1-t)+t^c) = (1-t)^{k-1} q(t)$,
where $q(1) = 1$.

Lemma

If $F(t) = (1-t)^{k-1}G(t)$*, where* $G(1) \neq 0$*, then the* i*'th derivative* $F^{(i)}(1) = 0$ *if* $i < k-1$ *and* $F^{(k-1)}(1) = (-1)^{k-1}G(1)$*.*

Now $\frac{d}{dt}\sum_{i=0}^{k-1}\sum_j (-1)^i \beta_{i,j} t^j = \sum_{i=0}^{k-1}\sum_j (-1)^i \beta_{i,j} j t^{j-1} = (1-t)^{k-2}q_1(t)$, where $q_1(1) = 1$. Substitute $t = 1$. Then multiply with t and derive again a.s.o.

Example

For $S = \langle 4, 7, 9 \rangle$ we get $(j = 1)$

$$0 - 16 - 18 - 21 + 25 + 30 = 0$$

and $(j = 2)$

$$-16^2 - 18^2 - 21^2 + 25^2 + 30^2 = (-1)^2 \cdot 2! \cdot 4 \cdot 7 \cdot 9.$$

A little larger example:

Let $S = \langle 8, 10, 12, 15 \rangle$. Then
$\beta_{1,20} = \beta_{1,24} = \beta_{1,30} = \beta_{2,44} = \beta_{2,50} = \beta_{2,54} = \beta_{3,74} = 1$.

For $j = 1$ we get $-20 - 24 - 30 + 44 + 50 + 54 - 74 = 0$,

for $j = 2$ we get $-20^2 - 24^2 - 30^2 + 44^2 + 50^2 + 54^2 - 74^2 = 0$,

and for $j = 3$ we get
$-20^3 - 24^3 - 30^3 + 44^3 + 50^3 + 54^3 - 74^3 = -3! \cdot 8 \cdot 10 \cdot 12 \cdot 15 = -86400$.

Let w_q be the number of generators divisible by q.

Theorem (Fel)

Let $S = \langle n_1, \ldots, n_k \rangle$. For every integers q and n such that $\gcd(n, q) = 1$, *and $w_q > 0$ we have*

$$\sum_{i=0}^{k-1} \sum_{j} (-1)^i \beta_{i,j} j^r e^{\frac{2n\pi i}{q} j} = 0$$

for $1 \leq r \leq w_q - 1$.

(A little improvement of Fel's result, he demanded $n \leq q/2$.)

Proof

Now $\sum_{i=0}^{k-1} \sum_j (-1)^i \beta_{i,j} t^j = (1 - t^q)^{w_q} q_1(t)$ and $e^{\frac{2\pi i}{q}}$ is a zero of multiplicity w_q of the RHS. Then the proof is the same.

Example

We use the same example as before, so let $S = \langle 8, 10, 12, 15 \rangle$. Then $\beta_{1,20} = \beta_{1,24} = \beta_{1,30} = \beta_{2,44} = \beta_{2,50} = \beta_{2,54} = \beta_{3,74} = 1$. Take $q = 3$. Then $w_q = 2$ (12 and 15 are divisible by 3.) We get the relations

$$-20e^{\frac{4\pi i}{3}} - 24 - 30 + 44e^{\frac{4\pi i}{3}} + 50e^{\frac{4\pi i}{3}} + 54 - 74e^{\frac{4\pi i}{3}} = 0$$

and

$$-20e^{\frac{2\pi i}{3}} - 24 - 30 + 44e^{\frac{2\pi i}{3}} + 50e^{\frac{2\pi i}{3}} + 54 - 74e^{\frac{2\pi i}{3}} = 0.$$

THANK YOU FOR YOUR ATTENTION!!!